THE OSCILLATORY BEHAVIOR OF SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT
The main objective of this paper is to study the oscillatory behavior of the solutions of the following non linear functional differential equations

\[ (a(t)\Psi(x(t))x'(t))' + p(t)x'(t) + q(t)f(x(g(t))) = 0. \]

The function \( f \) is not required to be monotonic.

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1. INTRODUCTION

In the last thirty years, many papers dealing with the oscillatory properties of nonlinear functional differential equations have appeared in the literature; see, for example [1–17] and the references therein. In most of these papers it is assumed that the equation under consideration has proper solutions and sufficient conditions under which these solutions are oscillatory are established. In the oscillation theory of nonlinear functional differential equations, one of the most important problems is to establish theorems involving comparison with linear equations. A search of the literature on this subject, i.e. the linearization of the second-order oscillation, shows that the most of the results obtained center around the Emden–Fowler equation, namely

\[ x''(t) + q(t)(|x(t)|^c)\text{sgn } x(t) = 0, \quad c > 0, \quad \frac{d}{dt}, \quad (1.1) \]

(see [18] and the references therein). Little is known regarding the oscillatory behavior of nonlinear equations of the type

\[ (a(t)\Psi(x(t))x'(t))' + p(t)x'(t) + q(t)f(x(g(t))) = 0, \quad (1.2) \]

via comparison with some linear second-order equations, particularly when the function \( f \) is not required to be monotonic. The purpose of this paper is to investigate the oscillatory behavior of the solutions of the second-order nonlinear functional differential Equation (1.2) where \( a : [t_0, \infty) \to (0, \infty), \quad p, q, g : [t_0, \infty) \to R = (-\infty, \infty), \quad \Psi, f : R \to R \) are continuous, \( a(t) > 0, \quad q(t) \geq 0 \) and not identically zero on any ray of the form \([t^*, \infty), \quad 0 \leq t_0 \leq t^* \) and \( xf(x) > 0 \) for \( x \neq 0 \). Moreover, \( g(t) \to \infty \) as \( t \to \infty, \quad 0 < c_1 \leq \Psi(x(.)) \leq c_2 \) for all \( x, c_1, c_2 > 0 \) and the functions \( a(\cdot), \quad q(\cdot), \quad g(\cdot) \) and \( \Psi(x(\cdot)) \) are continuously differentiable. By a solution of Equation (1.2), we mean a function \( x(.) : [T_0, \infty) \to R \) such that \( x(.) \) is continuously differentiable and satisfies Equation (1.2) for all sufficiently large \( t > T_0 \). A solution of Equation (1.2) is said to be oscillatory if it has an infinite sequence of zeros tending to infinity; otherwise a solution is said to be nonoscillatory. The Equation (1.2) is said to be oscillatory if all its solutions are oscillatory. We shall relate the oscillation problem of Equation (1.2) to that of some linear second-order ordinary differential equation. We would like to point out that the oscillatory behavior of Equation (1.2) has been discussed when this equation is either superlinear or sublinear (see [19, 20]).

2. THE MAIN RESULTS

In this section, we establish some oscillation theorems for Equation (1.2) via comparison with related linear equations. We present some notations which are needed in this section. We assume that:

\[ h(t) = \min\{t, g(t)\} \quad \text{and} \quad h'(t) > 0 \quad \text{for} \quad t \geq t_0. \quad (2.1) \]

For \( T \geq t_0 \) and for all \( t > T \), we assume that the functions \( R_1(t, T) \) and \( l(t) \) are given by

\[ R_1(t, T) = \int_T^t \frac{1}{a(s)} ds \quad \text{and} \quad l(t) = \frac{\Psi(x(t))a(h(t))}{h'(t)}. \]

Define

\[ R_{t_0} = \begin{cases} (-\infty, -t_0] \cup [0, \infty) & \text{if } t_0 > 0, \\ (-\infty, 0) \cup (0, \infty) & \text{if } t_0 = 0. \end{cases} \]
Let $C(R) = \{f : R \to R : f$ is continuous and $xf(x) > 0, x \neq 0\}$ and $C_B(R_{t_0}) = \{f \in C(R) : f$ is of bounded variation on any interval $[a, b] \subset R_{t_0}\}$.

**Lemma 2.1.** (see [21]) Suppose $t_0 \geq 0$ and $f \in C(R)$. Then $f \in C_B(R_{t_0})$ if and only if $f(x) = G(x)H(x)$ for all $x \in R_{t_0}$, where $G : R_{t_0} \to (0, \infty)$ is non-decreasing on $(-\infty, -t_0)$ and non increasing on $(t_0, \infty)$, and $H : R_{t_0} \to R$ is non-decreasing on $R_{t_0}$.

**Lemma 2.2.** Let $p(t) \geq 0$ and $q(t)$ be nonnegative and not identically zero on any ray of the form $[t^*, \infty)$ for $t^* \geq t_0$. Moreover we assume that
\[
\int_{t_0}^{\infty} \frac{1}{a(s)} \exp \left( \int_{t_0}^{s} -\frac{p(\tau)}{a(\tau)} \, d\tau \right) \, ds = \infty.
\] (2.2)

Then, if $x(t)$ is a nonoscillatory solution of Equation (1.2), we have $x(t)x'(t) > 0$ for all large $t$.

**Proof.** Let $x(t)$ be a nonoscillatory solution of Equation (1.2), and assume $x(t) > 0$ for $0 \leq t_0 \leq t$. If $x'(t_1) = 0$ and $q(t_1) > 0$ for some $t_1 \geq t_0$, then
\[
(a(t)\Psi(x(t))x'(t))|_{t=t_1} = -q(t_1)f(x(g(t_1))) < 0.
\] (2.3)

Thus we can prove that $x'(t)$ cannot have another zero after it vanishes once. Hence $x'(t)$ has a fixed sign for all sufficiently large $t$. Let $x'(t) < 0$ for $t_1 \leq t_2 \leq t$, then
\[
u'(t) + \frac{p(t)}{a(t)}u(t) \geq 0, \quad t \geq t_2,
\] (2.4)

where $u(t) = -a(t)\Psi(x(t))x'(t)$.

By integrating (2.4) from $t_2$ to $t$, we obtain
\[
x'(t)\Psi(x(t)) \leq -\frac{1}{a(t)}u(t_2) \exp \left( -\int_{t_2}^{t} \frac{p(\tau)}{a(\tau)} \, d\tau \right).
\] (2.5)

Integrating (2.5) and using (2.2) we get a contradiction. Thus the proof of Lemma 2.2 is completed. □

**Theorem 2.1.** Suppose $f \in C(R_{t_0})$ for $t_0 \geq 0$. Let $G$ and $H$ be a pair of continuous components of $f$ with $H$ being a non-decreasing one. Assume that the conditions (2.1) and (2.2) hold. Moreover, assume that, for some $k > 0$,
\[
\frac{H(x)}{x} \geq k, \quad x \neq 0.
\] (2.6)

If for every constant $k_1 \geq 1$ and for all large $T$ with $G(t) > T$, the linear equation
\[
(l(t)g'(t) + kq(t)G(k_1R_1(g(t), T))y(t) = 0
\] (2.7)
is oscillatory, then Equation (1.2) is oscillatory.

**Proof.** Let $x(t)$ be a nonoscillatory solution of Equation (1.2) and assume that $x(t) > 0$, $x(g(t)) > 0$ for $0 \leq t_0 \leq t$. 

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Using Lemma 2.2 and the well-known lemma 2. (see \cite[page 65]{22}), there exist \( t_1 \geq t_0 \) such that
\[
x(t) > 0, \ x'(t) > 0 \text{ and } (a(t)\Psi(x(t))x'(t))' \leq 0,
\]
for \( t \geq t_1 \). From the condition (2.1) and the fact that the function \( a(t)\Psi(x(t))x'(t) \) is positive and non increasing function on \([t_1, \infty)\), it follows that there exist positive constants \( b, d \), and \( t_2 \geq t_1 \) such that for \( t \geq t_2 \), we have
\[
x'(t) \leq \frac{a(h(t))}{a(t)} x'(h(t)), \quad \text{(2.9)}
\]
and
\[
x(h(t)) \geq b, \quad \text{(2.10)}
\]
By integrating (2.11) from \( t_2 \) to \( t \), we conclude that
\[
x(t) \leq x(t_2) + d \int_{t_2}^{t} \frac{1}{a(s)} ds \leq d_1 R_1(t, t_2)
\]
for some constant \( d_1 > 0 \).

Choose \( T \geq t_2 \) and a positive constant \( d^* \) such that \( t_2 \leq h(t) \leq g(t) \) and
\[
x(g(t)) \leq d^* R_1(g(t), t_2) \text{ and } x(h(t)) \leq d^* R_1(h(t), t_2), \text{ } t \geq T.
\]
Define \( w_1 \) by
\[
w_1(t) = -\frac{a(t)\Psi(x(t))x'(t)}{x(h(t))}, \quad t \geq T.
\]
On substituting (2.13) into the Equation (1.2) we deduce for \( t \geq T \), that
\[
(a(t)\Psi(x(t))x'(t))' = -(w_1(t)x(h(t)))' = -[w_1'(t)x(h(t)) + w_1(t)x'(h(t))h'(t)].
\]
Then
\[
-[w_1'(t)x(h(t)) + w_1(t)x'(h(t))h'(t)] + p(t)x'(t) + q(t)f(x(g(t))) = 0,
\]
and
\[
-w_1'(t)x(h(t)) + \frac{a(t)\Psi(x(t))x'(t)x'(h(t))h'(t)}{x(h(t))} + p(t)x'(t) + q(t)f(x(g(t))) = 0.
\]
Consequently,
\[
w_1'(t) = q(t) \frac{f(x(g(t)))}{x(h(t))} + \frac{p(t)x'(t)}{x(h(t))} + \frac{a(t)x'(t)h'(t)\Psi(x(t))x'(h(t))}{x^2(h(t))}.
\]
Now
\[ w'_1(t) = q(t)Q(t) + \frac{1}{S(t)}w^2_1(t), \quad (2.14) \]
where
\[ Q(t) = \frac{G(x(g(t)))H(x(g(t))) + M(t)x'(t)}{x(h(t))}, \quad M(t) = \frac{p(t)}{q(t)}, \quad (2.15) \]
and
\[ S(t) = \frac{a(t)\Psi(x(t))x'(t)}{x'(h(t))h'(t)}. \quad (2.16) \]

The Ricatti Equation (2.14) has a solution on \([T, \infty)\). It is well-known that this is equivalent to the nonoscillation of the linear equation
\[ (S(t)u'(t))' + q(t)Q(t)u(t) = 0, \quad t \geq T. \quad (2.17) \]

On substituting (2.1), (2.6), (2.9), and (2.12) into (2.15) and (2.16), we have
\[ S(t) \leq a(t)\Psi(x(t))a(h(t))a'(h(t)) \quad \text{and} \quad S(t) = l(t), \quad t \geq T, \quad (2.18) \]

and
\[ Q(t) \geq G(d^*R_1(g(t), t_2)H(x(h(t)))) + M(t)x'(t) \quad \text{and} \quad Q(t) \geq kG(d^*R_1(g(t), t_2)), \quad t \geq T. \quad (2.19) \]

Thus
\[ Q(t) \geq kG(d^*R_1(g(t), t_2)), \quad t \geq T. \quad (2.19) \]

Then, an application of the Picone–Sturm comparison theorem (see [23]) to Equation (2.17) yields the nonoscillation of the linear equation
\[ (l(t)y'(t))' + kG(d^*R_1(g(t), t_2))q(t)y = 0. \]

This contradicts the hypothesis that the Equation (2.7) is oscillatory. Then the Equation (1.2) is oscillatory and the proof of Theorem 2.1 is completed. \[ \square \]

**Corollary 2.1.** Suppose \( f \in C(R_{t_0}) \) for \( t_0 \geq 0 \). Let \( G \) and \( H \) be a pair of components of \( f \) with \( H \) being the non-decreasing one. Assume that the conditions (2.1), (2.2), and (2.6) hold. Moreover, assume for every constant \( k_1 \geq 1 \) and all large \( T \) with \( g(t) \geq T \) that the inequality
\[ \lim_{t \to \infty} \inf \left[ \int_T^t \frac{1}{l(s)} ds \right] \left[ \int_t^\infty G(k_1R_1(g(u), T))q(u)du \right] > \frac{1}{4k}, \quad (2.20) \]
holds, for some \( k > 0 \), which is the same constant in Theorem 2.1. Then Equation (1.2) is oscillatory.
Theorem 2.2. Suppose $f \in C(R_{+})$ for $t_{0} \geq 0$. Let $G$ and $H$ be a pair of continuous components of $f$ with $H$ being the non-decreasing one. Moreover, assume that conditions (2.1) and (2.2) hold. Further, assume also that the following condition

$$H(x) \text{sgn } x \geq |x|^{c}, \ x \neq 0,$$

with $c > 1$ holds. If for every value of constants $k_{1} \geq 1$ and $k_{2} > 0$, and for all large $T$ with $g(t) > T$, the linear equation

$$(l(t)y'(t))' + k_{2}G(k_{1}R_{1}(g(t), T))q(t)y(t) = 0,$$

is oscillatory, then Equation (1.2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (1.2), say $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_{0}$. Following the same way as in the proof of Theorem 2.1 and for all $t \geq T$, we obtain the conditions (2.14)–(2.17) which are satisfied. Using (2.1), (2.10), and (2.21) in (2.15) we have

$$Q(t) \geq G(d^{*}R_{1}(g(t), t_{2}))(x(h(t)))^{c-1} + M(t) \frac{x'(t)}{x(h(t))}.$$

Thus

$$Q(t) \geq b^{-1}G(d^{*}R_{1}(g(t), t_{2})), \ t \geq T.$$

With reference to the Picone–Sturm comparison theorem and using (2.17), (2.23) we obtain the nonoscillation of the linear equation

$$(l(t)y'(t))' + b^{-1}G(d^{*}R_{1}(g(t), t_{2}))q(t)y(t) = 0.$$

This contradicts the hypothesis that Equation (2.22) is oscillatory. Thus Equation (1.2) is oscillatory and the proof of Theorem 2.2 is completed. □

Theorem 2.3. Suppose $f \in C(R_{+})$ for $t_{0} \geq 0$. Let $G$ and $H$ be a pair of continuous components of $f$ with $H$ being the non-decreasing one. Moreover, assume that the conditions (2.1), (2.2), and (2.21) hold with $0 < c < 1$. If the linear equation

$$(l(t)y'(t))' + k_{2}G(k_{1}R_{1}(g(t), T))R_{1}(h(t), T))^{c-1}q(t)y(t) = 0,$$

is oscillatory for every constants $k_{1} \geq 1, 0 \leq k_{2} \leq 1$ and for all large $T$ with $h(t) > T$, then Equation (1.2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (1.2), say $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_{0}$. Following the same way as in the proof of Theorem 2.1 and for all $t \geq T$, we see that the conditions (2.14)–(2.17) are satisfied. On substituting (2.1), (2.12), and (2.22) into (2.15), we have

$$Q(t) \geq G(d^{*}R_{1}(g(t), t_{2}))x(h(t))^{c-1} + M(t) \frac{x'(t)}{x(h(t))} \geq G(d^{*}R_{1}(g(t), t_{2}))(d^{*})^{c-1}(R_{1}(h(t), t_{2}))^{c-1} + M(t) \frac{x'(t)}{x(h(t))},$$
and consequently, we get
\[ Q(t) \geq G(d^* R_1(g(t), t_2))(d^*)^{c-1}(R_1(h(t), t_2))^{c-1}, \ t \geq T. \]  

(2.25)

With reference to the Picone–Sturm comparison theorem and using (2.17), (2.25) we obtain the nonoscillation of the linear equation
\[ (l(t)y'(t))' + q(t)(d^*)^{c-1}G(d^* R_1(g(t), t_2))(R_1(h(t), t_2))^{c-1}y(t) = 0. \]

This contradicts the hypothesis that Equation (2.24) is oscillatory. Then the Equation (1.2) is oscillatory and the proof of Theorem 2.3 is completed.

An application of Theorems 2.1–2.3, we present a linearization result of a special case of Equation (1.2) namely
\[ (a(t)\Psi(x(t))x'(t))' + p(t)x'(t) + q(t)(|x(g(t))|^c)sgn x(g(t)) = 0, \ c > 0. \]  

(2.26)

**Corollary 2.2.** Let the conditions (2.1) and (2.2) hold. The Equation (2.26) is oscillatory if for all large \( T \) with \( h(t) > T \), the linear equation
\[ (l(t)y'(t))' + Q(t, T)q(t)y(t) = 0, \]

is oscillatory, where
\[
Q(t, T) = \begin{cases} 
  k_3, & \text{where } k_3 \text{ is any positive constant} \quad \text{if } c > 1, \\
  1, & \text{if } c = 1, \\
  k_4(R_1(h(t), T))^{c-1}, & \text{if } 0 < k_4 \leq 1, \quad \text{if } 0 < c < 1.
\end{cases}
\]

**Remark 2.1.** If we set \( p(t) = 0 \) and \( \Psi(x(t)) = 1 \) in our main results, we obtain the oscillatory behavior of the solution of following equation (see [24])
\[ (a(t)x'(t))' + q(t)f(x(g(t))) = 0. \]  

(2.27)

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