On the Rational Recursive Sequence

\[ x_{n+1} = Ax_n + (\beta x_n + \gamma x_{n-k}) / (Bx_n + Cx_{n-k}) \]

E. M. E. Zayed 1 and M. A. El-Moneam
Zagazig University
Department of Mathematics
Faculty of Science
Zagazig, Egypt
emezayed@hotmail.com
mabdelmeneam2004@yahoo.com

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Abstract

The main objective of this paper is to study some qualitative behavior of the solutions of the difference equation

\[ x_{n+1} = Ax_n + (\beta x_n + \gamma x_{n-k}) / (Bx_n + Cx_{n-k}) , \quad n = 0, 1, 2, \ldots \]

where the initial conditions \( x_{-k}, \ldots, x_{-1}, x_0 \) are arbitrary positive real numbers and the coefficients \( A, B, C, \beta \) and \( \gamma \) are positive constants, while \( k \) is a positive integer number. Some numerical examples will be given to illustrate our results.

Keywords: Difference equations, Prime period two solution, Locally asymptotically stable, Global attractor, Global stability.

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1Present address: Department of Mathematics, Faculty of Science, Taif University, El-Taif, EL-Hawiyah, P.O. Box 888, Kingdom of Saudi Arabia
1 Introduction

Our goal in this paper is to investigate some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = Ax_n + \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

(1)

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients $A, B, C, \beta$ and $\gamma$ are positive constants, while $k$ is a positive integer number. The global stability of Eq. (1) for $A = 0$ and $k = 1$ has been investigated in [3]. The global stability of Eq. (1) for $A = 0$ has been investigated in [6].

Dehghan et al. [1] investigated the global stability and the boundedness character of the equation

$$x_{n+1} = \frac{x_n + p}{x_n + qx_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where the parameters $p$ and $q$ and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are positive real numbers, $k = \{1, 2, 3, \ldots\}$.

Li and Sun [4] investigated the periodic character and the global stability of all positive solutions of the equation

$$x_{n+1} = \frac{px_n + x_{n-k}}{q + x_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where the parameters $p$ and $q$ and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are positive real numbers, $k = \{1, 2, 3, \ldots\}$.

Devault et al. [2] investigated the periodic character and the global stability of solutions of the equation

$$x_{n+1} = \frac{p + x_{n-k}}{q + x_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where the parameters $p$ and $q$ and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are positive real numbers, $k = \{1, 2, 3, \ldots\}$.

M. Saleh et al. [6] investigated the periodic character and the global stability of all positive solutions of the equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where the parameters $\beta, \gamma$ and $B, C$ and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are positive real numbers, $k = \{1, 2, 3, \ldots\}$.

Our interest now is to study the behavior of solutions of Eq. (1) in the general case where $A \neq 0$ and $k$ is a positive integer number. For the related work see [7–29]. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. Note that Eq. (1) can be considered as a generalization of that obtained in [3,6].
Definition 1.1 A difference equation of order \((k+1)\) is of the form
\[ x_{n+1} = F(x_n, x_{n-k}), \quad n = 0, 1, 2, \ldots \]  
(2)
where \(F\) is a continuous function which maps some set \(J^{k+1}\) into \(J\) where \(J\) is a set of real numbers. An equilibrium point \(\tilde{x}\) of this equation is a point that satisfies the condition \(\tilde{x} = F(\tilde{x}, \tilde{x})\). That is, the constant sequence \(\{x_n\}_{n=-k}^{\infty} = \tilde{x}\) for all \(n \geq -k\) is a solution of that equation.

Definition 1.2 Let \(\tilde{x} \in (0, \infty)\) be an equilibrium point of the difference equation (2). Then
(i) An equilibrium point \(\tilde{x}\) of the difference equation (2) is called locally stable if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that, if \(x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)\) with \(|x_{-k} - \tilde{x}| + \ldots + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \delta\), then \(|x_n - \tilde{x}| < \varepsilon\) for all \(n \geq -k\).

(ii) An equilibrium point \(\tilde{x}\) of the difference equation (2) is called locally asymptotically stable if it is locally stable and there exists \(\gamma > 0\) such that, if \(x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)\) with \(|x_{-k} - \tilde{x}| + \ldots + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \gamma\), then
\[ \lim_{n \to \infty} x_n = \tilde{x}. \]

(iii) An equilibrium point \(\tilde{x}\) of the difference equation (2) is called a global attractor if for every \(x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)\) we have
\[ \lim_{n \to \infty} x_n = \tilde{x}. \]

(iv) An equilibrium point \(\tilde{x}\) of the equation (2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point \(\tilde{x}\) of the difference equation (2) is called unstable if it is not locally stable.

Definition 1.3 A sequence \(\{x_n\}_{n=-k}^{\infty}\) is said to be periodic with period \(p\) if \(x_{n+p} = x_n\) for all \(n \geq -k\). A sequence \(\{x_n\}_{n=-k}^{\infty}\) is said to be periodic with prime period \(p\) if \(p\) is the smallest positive integer having this property.

Definition 1.4 A positive semi-cycle of \(\{x_n\}_{n=-k}^{\infty}\) consists of "a string" of terms \(\{x_l, x_{l+1}, \ldots, x_m\}\) all greater than or equal to \(\tilde{x}\), with \(l \geq -k\) and \(m \leq \infty\) such that
\[ \text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} < \tilde{x}, \]
and
\[ \text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} < \tilde{x}. \]

A negative semi-cycle of \(\{x_n\}_{n=-k}^{\infty}\) consists of "a string" of terms \(\{x_l, x_{l+1}, \ldots, x_m\}\) all less than \(\tilde{x}\), with \(l \geq -k\) and \(m \leq \infty\) such that
\[ \text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} \geq \tilde{x}, \]
and
\[ \text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} \geq \tilde{x}. \]
Definition 1.5 Eq. (2) is said to be permanent if there exist positive real numbers \( m \) and \( M \) such that for every solution \( \{x_n\}_{n=-k}^\infty \) of Eq. (2) there exists a positive integer \( N \geq -k \) which depends on the initial conditions, such that

\[
m \leq x_n \leq M, \quad \text{for all } n \geq N.
\]

The linearized equation of the difference equation (2) about the equilibrium point \( \tilde{x} \) is the linear difference equation

\[
y_{n+1} = \frac{\partial F(\tilde{x}, \tilde{x})}{\partial x_n} y_n + \frac{\partial F(\tilde{x}, \tilde{x})}{\partial x_{n-k}} y_{n-k}.
\]

Now, assume that the characteristic equation associated with (3) is

\[
p(\lambda) = \lambda^{k+1} - p_0 \lambda^k - p_1 = 0,
\]

where

\[
p_0 = \frac{\partial F(\tilde{x}, \tilde{x})}{\partial x_n}, \quad p_1 = \frac{\partial F(\tilde{x}, \tilde{x})}{\partial x_{n-k}}.
\]

Theorem 1.1 ([5]). The linearized stability theorem.

Suppose \( F \) is a continuously differentiable function defined on an open neighbourhood of the equilibrium \( \tilde{x} \). Then the following statements are true.

(i) If all the roots of the characteristic equation (4) of the linearized equation (3) have absolute value less than one, then the equilibrium point \( \tilde{x} \) of Eq. (2) is locally asymptotically stable.

(ii) If at least one root of Eq. (4) has absolute value greater than one, then the equilibrium point \( \tilde{x} \) of Eq. (2) is not locally stable.

(iii) If all the roots of Eq. (4) have absolute value greater than one, then the equilibrium point \( \tilde{x} \) of Eq. (2) is a source.

1.1 Change of variables

By using the change of variables \( x_n = \gamma C y_n \). Then Eq. (1) reduces to the difference equation

\[
y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}}, \quad n = 0, 1, 2, \ldots.
\]

where \( p = \frac{p}{2} \) and \( q = \frac{q}{2} \), with \( p, q \in (0, \infty) \), \( y_{-k}, \ldots, y_{-1}, y_0 \in (0, \infty) \). To avoid a degenerate situation we also assume that \( p \neq q \).

Next, we investigate the equilibrium points of the nonlinear rational difference equation (5) where the parameters \( p, q \) and the initial conditions \( y_{-k}, \ldots, y_{-1}, y_0 \) are arbitrary positive real numbers, while \( k \) is a positive integer number.

The equilibrium points of Eq. (5) are the positive solutions of the equation

\[
\tilde{y} = Ay + \frac{py + \tilde{y}}{qy + \tilde{y}}.
\]

If \( 0 < A < 1 \), then the only positive equilibrium point is

\[
\tilde{y} = \frac{p + 1}{(1 - A)(q + 1)}.
\]
1.2 Linearization

In this section, we derive the linearized equation of Eq.(5). To this end, we introduce a continuous function $F : (0, \infty)^2 \to (0, \infty)$ which is defined by

$$F(u_0, u_1) = Au_0 + \frac{pu_0 + u_1}{qu_0 + u_1}.$$ (7)

Therefore,

$$\begin{align*}
\frac{\partial F(u_0, u_1)}{\partial u_0} &= A + \frac{(p-q)u_1}{(qu_0 + u_1)^2}, \\
\frac{\partial F(u_0, u_1)}{\partial u_1} &= -\frac{(p-q)u_0}{(qu_0 + u_1)^2}. 
\end{align*}$$ (8)

From (6) and (8) we have

$$\begin{align*}
\frac{\partial F(\tilde{y}, \tilde{y})}{\partial u_0} &= A + \frac{(p-q)(1-A)}{(p+1)(q+1)} = \rho_0, \\
\frac{\partial F(\tilde{y}, \tilde{y})}{\partial u_1} &= -\frac{(p-q)(1-A)}{(p+1)(q+1)} = \rho_1. 
\end{align*}$$ (9)

Then the linearized equation of the difference equation (5) about $\tilde{y}$ is

$$y_{n+1} - \rho_0 y_n - \rho_1 y_{n-k} = 0,$$ (10)

where $\rho_0$ and $\rho_1$ are given by (9).

**Theorem 1.2** ([3]). Assume that $\rho_0, \rho_1 \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Then

$$|\rho_0| + |\rho_1| < 1,$$ (11)

is a sufficient condition for the asymptotic stability of the difference equation (5). Suppose in addition that one of the following two cases holds:
(i) $k$ is an odd integer and $\rho_1 > 0$.
(ii) $k$ is an even integer and $\rho_0 \rho_1 > 0$.

Then (11) is also a necessary condition for the asymptotic stability of Eq.(5).

**Theorem 1.3** ([5, p.18]). Let $F : [a, b]^{k+1} \to [a, b]$ be a continuous function, where $k$ is a positive integer, and where $[a, b]$ is an interval of real numbers and consider the difference equation (2). Suppose that $F$ satisfies the following conditions:
(i) For each integer $i$ with $1 \leq i \leq k+1$, the function $F(z_1, z_2, \ldots, z_{k+1})$ is weakly monotonic in $z_i$ for fixed $z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{k+1}$.
(ii) If $(m, M)$ is a solution of the system

$$m = F(m_1, m_2, \ldots, m_{k+1}) \quad \text{and} \quad M = F(M_1, M_2, \ldots, M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, \ldots, k+1$, we set

$$m_i = \begin{cases} 
    m & \text{if } F \text{ nondecreasing in } z_i \\
    M & \text{if } F \text{ nonincreasing in } z_i 
\end{cases}$$
and

\[ M_i = \begin{cases} 
M & \text{if } F \text{ nondecreasing in } z_i \\
m & \text{if } F \text{ nonincreasing in } z_i.
\end{cases} \]

Then there exists exactly one equilibrium point \( \tilde{x} \) of the difference equation (2), and every solution of (2) converges to \( \tilde{x} \).

## 2 Semi-cycle analysis

**Theorem 2.1** Assume that \( F \in [(0, \infty)^2, (0, \infty)] \) is a continuous function such that \( F(x, y) \) is increasing (respectively, decreasing) in \( x \) for each fixed \( y \), and \( F(x, y) \) is decreasing (respectively, increasing) in \( y \) for each fixed \( x \). Let \( \tilde{y} \) be a positive equilibrium of Eq.(5). Then except possibly for the first semi-cycle, every oscillatory solution of Eq.(5) has semi-cycle of length at least \( k \).

**Proof:** When \( k = 1 \), the proof is presented as Theorem 1.7.1 in [3]. We just give the proof of the theorem 4 for \( k = 2 \). The proof of the theorem 4 for \( k \geq 3 \), is similar and omitted here. Let \( \{y_n\} \) be a solution of Eq.(5) with at least three semi-cycles. Then, there exists \( N \geq 0 \) such that either

\[ y_{N-1} < \tilde{y} \leq y_{N+1}, \]

or

\[ y_{N-1} \geq \tilde{y} > y_{N+1}. \]

We first assume that

\[ y_{N-1} < \tilde{y} \leq y_{N+1}. \]

Since the function \( F(x, y) \) is increasing (respectively, decreasing) in \( x \) for each fixed \( y \), then we have

\[ y_{N+2} = F(y_{N+1}, y_{N-1}) < F(\tilde{y}, \tilde{y}) = \tilde{y}, \]

and

\[ y_{N+3} = F(y_{N+2}, y_{N}) > F(\tilde{y}, y_{N}). \]

Since the function \( F(x, y) \) is decreasing in \( y \) for fixed \( x \), then

\[ F(\tilde{y}, y_{N}) > F(\tilde{y}, \tilde{y}) = \tilde{y} \quad \text{for} \quad y_{N} < \tilde{y}. \]

Hence, we obtain

\[ y_{N+2} < \tilde{y} < y_{N+3}. \]

Similarly, we can prove the theorem if \( y_{N-1} \geq \tilde{y} > y_{N+1} \) which is omitted here. Now, the proof of Theorem 2.1 is completed. \( \Box \)
3 Local stability

In this section, we investigate the local stability of the positive solutions of Eq.(5). By using Theorems 1.1 and 1.2, we have the following result.

**Theorem 3.1**

(i) Assume that

\[
0 < (p - q) < \frac{1}{2} (p + 1) (q + 1) \quad \text{and} \quad 0 < A < 1,
\]

then, the positive equilibrium point \( \tilde{y} \) of Eq.(5) is locally asymptotically stable.

(ii) If \( k \) is either odd or even, \( p < q \), \( 0 < A < 1 \) and

\[
A > \frac{(q - p) (1 - A)}{(p + 1) (q + 1)}.
\]

Then (11) is the necessary and sufficient condition for the asymptotically stable of Eq.(5).

**Proof:** Under the assumptions of part (i), we deduce from (9) that

\[
|\rho_0| + |\rho_1| = \left| A + \frac{(p - q) (1 - A)}{(p + 1) (q + 1)} \right| + \left| - \frac{(p - q) (1 - A)}{(p + 1) (q + 1)} \right|
\]

\[
= A + \frac{(p - q) (1 - A)}{(p + 1) (q + 1)} + \frac{(p - q) (1 - A)}{(p + 1) (q + 1)}
\]

\[
= \frac{A (p + 1) (q + 1) + 2 (p - q) (1 - A)}{(p + 1) (q + 1)} < 1.
\]

According to Theorem 1.2, the proof of part (i) of Theorem 3.1 is completed.

Under the assumptions of part (ii), we deduce from (9) that

\[
|\rho_0| + |\rho_1| = \left| A - \frac{(q - p) (1 - A)}{(p + 1) (q + 1)} \right| + \left| \frac{(q - p) (1 - A)}{(p + 1) (q + 1)} \right|
\]

\[
= A - \frac{(q - p) (1 - A)}{(p + 1) (q + 1)} + \frac{(q - p) (1 - A)}{(p + 1) (q + 1)}
\]

\[
= A < 1.
\]

Furthermore, if \( k \) is an odd positive integer, we get

\[
\rho_1 = \frac{(q - p) (1 - A)}{(p + 1) (q + 1)} > 0,
\]

while, if \( k \) is an even positive integer, we have

\[
\rho_0 \rho_1 = \left( A - \frac{(q - p) (1 - A)}{(p + 1) (q + 1)} \right) \left( \frac{(q - p) (1 - A)}{(p + 1) (q + 1)} \right) > 0.
\]

According to Theorem 1.2, the proof of part (ii) of Theorem 3.1 is completed. Thus, the proof of Theorem 3.1 is now finished. \( \square \)
4 Periodic solutions

In this section, we investigate the periodic character of the positive solutions of Eq.(5).

**Theorem 4.1**

(1) If \( p > q \), then Eq.(5) has no positive solutions of prime period two.

(2) If \( k \) is an even integer, then Eq.(5) has no positive solutions of prime period two.

(3) If \( k \) is an odd integer, then Eq.(5) has prime period two solutions

\[ \ldots, \Phi, \Psi, \Phi, \Psi, \ldots \]

if the following condition is valid:

\[
(p - 1) (q - 1) (A + 1) > -4 (qA + p),
\]

where \( p < 1 \) and \( q > 1 \) while the values of \( \Phi \) and \( \Psi \) are the (positive and distinct) solutions of the quadratic equation

\[
t^2 - \frac{(1 - p)}{(qA + 1)} t + \frac{(qA + p) (1 - p)}{(q - 1) (A + 1) (qA + 1)^2} = 0.
\]

**Proof:** First of all, we prove the part (1) in the case \( p > q \). Assume for the sake of contradiction that there exists distinctive positive real numbers \( \Phi \) and \( \Psi \), such that

\[ \ldots, \Phi, \Psi, \Phi, \Psi, \ldots \]

is a prime period two solution of Eq.(5).

If \( k \) is odd, then \( y_{n+1} = y_{n-k} \). It follows from Eq.(5) that

\[
\Phi = A \Psi + \frac{p \Phi + \Phi}{q \Psi + \Phi} \quad \text{and} \quad \Psi = A \Phi + \frac{p \Phi + \Psi}{q \Phi + \Psi}
\]

Consequently, we obtain

\[
q \Phi \Psi + \Phi^2 = qA \Psi^2 + A \Phi \Psi + p \Psi + \Phi, \quad (12)
\]

and

\[
q \Phi \Psi + \Psi^2 = qA \Phi^2 + A \Phi \Psi + p \Phi + \Psi. \quad (13)
\]

By subtracting (13) from (12), we deduce that

\[
\Phi + \Psi = \frac{1 - p}{qA + 1}, \quad (14)
\]

while, by adding (12), (13) and using (14) we get

\[
\Phi \Psi = \frac{(qA + p) (1 - p)}{(q - 1) (A + 1) (qA + 1)^2}. \quad (15)
\]
Consequently, we conclude from (15) that

\[(qA + p) (1 - p) > 0 \quad \text{and} \quad (q - 1) (A + 1) (qA + 1)^2 > 0 \]  \hspace{1cm} (16)

or

\[(qA + p) (1 - p) < 0 \quad \text{and} \quad (q - 1) (A + 1) (qA + 1)^2 < 0. \]  \hspace{1cm} (17)

From (16) we deduce that \(q > 1\). Since we have \(p > q\) then \(p > 1\) and hence \((1 - p) < 0\). Thus, we deduce that \(\Phi \Psi < 0\). This contradicts the hypothesis that \(\Phi \Psi > 0\). Also, from (17) we have \(p > 1\) and hence \((1 - p) < 0\). Thus, we deduce that \(\Phi + \Psi < 0\). This contradicts the hypothesis that \(\Phi + \Psi > 0\). Hence the proof of part (1) is completed.

(2) Assume for the sake of contradiction that there exists distinctive positive real numbers \(\Phi\) and \(\Psi\), such that

\[\ldots, \Phi, \Psi, \Phi, \Psi, \ldots.\]

is a prime period two solution of Eq.(5). If \(k\) is even, then \(y_n = y_{n-k}\). It follows from the difference equation (5) that

\[\Phi = A\Psi + \frac{p\Psi + \Phi}{q\Psi + \Phi} \quad \text{and} \quad \Psi = A\Phi + \frac{p\Phi + \Psi}{q\Phi + \Psi}.\]

Hence we have \((\Phi - \Psi) (A + 1) = 0\). Thus \(\Phi = \Psi\). This contradicts the hypothesis that \(\Phi\) and \(\Psi\) distinct positive real number. Thus, the proof of part (2) is completed.

(3) Assume that Eq.(5) has prime period two solutions

\[\ldots, \Phi, \Psi, \Phi, \Psi, \ldots.\]

If \(k\) is odd, then \(y_{n+1} = y_{n-k}\). It follows from Eq.(5) that

\[\Phi = A\Psi + \frac{p\Psi + \Phi}{q\Psi + \Phi} \quad \text{and} \quad \Psi = A\Phi + \frac{p\Phi + \Psi}{q\Phi + \Psi}.\]

Then we have

\[\Phi + \Psi = \frac{1 - p}{qA + 1} \quad \text{and} \quad \Phi\Psi = \frac{(qA + p) (1 - p)}{(q - 1) (A + 1) (qA + 1)^2}.\]

Now, we consider the quadratic equation

\[t^2 - \frac{(1 - p)}{(qA + 1)} t + \frac{(qA + p) (1 - p)}{(q - 1) (A + 1) (qA + 1)^2} = 0.\]

So, the values of \(\Phi\) and \(\Psi\) are the (positive and distinct) solutions of the above quadratic equation. Thus, we get

\[t = \frac{(1 - p) \pm \delta}{2 (qA + 1)},\]

where

\[\delta = \sqrt{(1 - p)^2 - 4 (qA + p) (1 - p) / (q - 1) (A + 1)}.\]
Thus, we deduce that
\[
\left( \frac{p - 1}{qA + 1} \right)^2 > -4(qA + p)(p - 1) \frac{1}{(q - 1)(A + 1)(qA + 1)^2},
\]
and hence, we have
\[
(p - 1)(q - 1)(A + 1) > -4(qA + p).
\]
Thus, the proof of part (3) is completed. The proof of Theorem 4.1 is now finished. 

5 Boundedness character

In this section, we investigate the boundedness character of the positive solutions of Eq.(5).

**Theorem 5.1** Let \( \{y_n\}_{n=-k}^\infty \) be a solution of Eq.(5). Then the following statements are true:

1. Suppose \( p < q \) and assume that for some \( N \geq 0 \)
   \[ y_{N-k+1}, \ldots, y_{N-1}, y_N \in \left[ \frac{p}{q}, 1 \right], \]
   then
   \[ y_n \in \left[ \frac{p}{q} \left( A + \frac{p + 1}{q + 1} \right), A + 1 \right], \quad \text{for all } n > N. \]

2. Suppose \( p > q \) and assume that for some \( N \geq 0 \)
   \[ y_{N-k+1}, \ldots, y_{N-1}, y_N \in \left[ 1, \frac{p}{q} \right], \]
   then
   \[ y_n \in \left[ A + 1, \frac{p}{q} \left( A + \frac{p + 1}{q + 1} \right) \right], \quad \text{for all } n > N. \]

**Proof:** First of all, if for some \( N > 0, \frac{p}{q} \leq y_N \leq 1 \) and \( p < q \), then
\[
y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \leq Ay_n + \frac{qy_n + y_{n-k}}{qy_n + y_{n-k}} = Ay_n + 1 \leq A + 1,
\]
and
\[
y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \geq \frac{p}{q} \left( A + \frac{p + 1}{q + 1} \right).
\]
Thus, the proof of part (1) is completed.

Secondly, if for some \( N > 0, 1 \leq y_N \leq \frac{p}{q} \) and \( p > q \), then
\[
y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \leq \frac{p}{q} \left( A + \frac{p + 1}{q + 1} \right),
\]
and
\[ y_{n+1} = Ay_n + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \geq Ay_n + \frac{py_n + y_{n-k}}{py_n + y_{n-k}} \geq Ay_n + 1 \geq A + 1. \]

Thus, the proof of part (2) is completed. The proof of Theorem 5.1 is now finished. \( \square \)

6 Global stability

In this section, we investigate the global stability of the positive solutions of Eq.(5).

**Theorem 6.1** Assume that \( p > q \) and \( 0 < A < 1 \), then, the positive equilibrium point \( \tilde{y} \) of Eq.(5) is globally asymptotically stable.

**Proof:** Under these assumptions, we have shown in part (i) of Theorem 3.1 that the positive equilibrium point \( \tilde{y} \) of Eq.(5) is locally asymptotically stable. It remains to prove that \( \tilde{y} \) is a global attractor. To this end, we consider the function
\[ F(x, y) = Ax + \frac{px + y}{qx + y}. \]

Since \( p > q \), then the function \( F(x, y) \) is increasing in \( x \) for each fixed \( y \), and decreasing in \( y \) for each fixed \( x \). Suppose that \((m, M)\) is a solution of the system
\[ M = F(M, m) \quad \text{and} \quad m = F(m, M). \]

Then we get
\[ M = AM + \frac{pM + m}{qM + m} \quad \text{and} \quad m = Am + \frac{pm + M}{qm + M}. \]

From which we have
\[ (1 - A) \quad Mm = pM + m - q(1 - A)M^2. \]  \hspace{1cm} (18)

and
\[ (1 - A) \quad Mm = pm + M - q(1 - A)m^2. \]  \hspace{1cm} (19)

From (18) and (19), we obtain
\[ (m - M) [p - 1 - q(1 - A)(m + M)] = 0. \]  \hspace{1cm} (20)

The relation (20) gives \( M = m \). According to Theorem 1.3, the proof of Theorem 6.1 is now completed. \( \square \)

**Theorem 6.2** If \( k \) is either even or odd, \( p < q \), \( 0 < A < 1 \) and \( A > \frac{(q-p)(1-A)}{(p+1)(q+1)} \), then, the positive equilibrium point \( \tilde{y} \) of Eq.(5) is globally asymptotically stable.
Proof: Under these assumptions, we have shown in part (ii) of Theorem 3.1 that the positive equilibrium point \( \tilde{y} \) of Eq. (5) is locally asymptotically stable. It remains to prove that \( \tilde{y} \) is a global attractor. To this end, we consider the function

\[
F(x, y) = Ax + \frac{px + y}{qx + y}.
\]

Since \( p < q \), then the function \( F(x, y) \) is decreasing in \( x \) for each fixed \( y \), and increasing in \( y \) for each fixed \( x \). Suppose that \( (m, M) \) is a solution of the system

\[
m = F(M, m) \quad \text{and} \quad M = F(m, M).
\]

Then we get

\[
m = AM + \frac{pM + m}{qM + m} \quad \text{and} \quad M = Am + \frac{pm + M}{qm + M}.
\]

From which we have

\[
(q - A) \, Mm = pm + M - M^2 + qMm^2. \tag{21}
\]

and

\[
(q - A) \, Mm = pM + m - m^2 + qAM^2. \tag{22}
\]

From (21) and (22), we obtain

\[
(m - M) \left[ p - 1 + (qA + 1) \, (m + M) \right] = 0. \tag{23}
\]

The relation (23) gives \( M = m \). According to Theorem 1.3, the proof of Theorem 6.2 is completed.

7 Numerical examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the nonlinear difference equation (5).

Example 1. Figure 1, shows that Eq.(5) has prime period two solution if \( k = 1, \, y_{-1} = 0.383, \, y_0 = 0.079, \, A = 0.01, \, p = 0.5, \, q = 20. \)
On the Rational Recursive Sequence

Figure 1: \( y_{n+1} = 0.01y_n + \frac{0.5y_n + y_{n-1}}{20y_n + y_{n-1}} \)

**Example 2.** Figure 2, shows that the solution of Eq.(5) has global stability if \( k = 1, \ p > q, \ y_{-1} = 1, \ y_0 = 2, \ A = 0.25, \ p = 300, \ q = 5. \)

Figure 2: \( y_{n+1} = 0.25y_n + \frac{300y_n + y_{n-1}}{5y_n + y_{n-1}} \)

**Example 3.** Figure 3, shows that the solution of Eq.(5) has global stability if \( k = 1, \ p < q, \ y_{-1} = 1, \ y_0 = 2, \ A = 0.25, \ p = 5, \ q = 300. \)

**Note that** example 1 verifies theorem 4.1(3) which show that Eq.(2) has prime period two solution. But example 2 verifies theorem 6.1 for \( p > q \) which shows that the solution of Eq.(2) is globally asymptotic stable, while example 3 verifies theorem 6.2 for \( p < q \) which shows that the solution of Eq.(2) is globally asymptotic stable.
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