# Linear Algebra (Math-324)
# Lecture Notes

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# Contents

## To Students

**Mid Term Exams:**
The first mid-term exam will be given during the 7th week. The second mid-term exam will be given during the 12th week.

**QUIZZES:**
There will be many quizzes (probably one almost every week period). Quizzes will relate to current and previous topics. A quiz may be given at any time during any class period – immediately after a lecture, at the beginning or end of a class, etc. There will be no make-up quizzes – none even later during the same class period. Quizzes will be given only to those students who are present when the quizzes are passed out.

## Preface

Linear Algebra is used in many areas as: engineering, biology, medicine, business, statistics, physics, mathematics, numerical analysis and humanities. The reason: many real world systems consist of many parts which interact linearly. Analysis of such systems involves the notions and the tools from Linear Algebra. Our main goal in writing these notes was to give to the student a concise overview of the main concepts, ideas and results that usually are covered in the first course on linear algebra for mathematicians. These notes should be viewed as a supplementary notes to a regular book for linear algebra.

| معجم المصطلحات | |
|---|---|
| Union | اتحاد |
| Linear Dependance | ارتباط خطي |
| Linear Independance | استقلال خطي |
| Dimensions | ابعاد |
| Trivial | تافه |
| Associative | تجميعي |
| Linear Transformation | تحويل خطي |
| Intersection | تقاطع |
| Direct Sum | جمع مباشر |
| Rank | رتبه |
| Nullity | صفرية |
| Image | صورة |
| Inner Product | ضرب داخلي |
| Orthogonal | عمودي |
| Subspace | فضاء جزئي |
| Vector Space | فضاء متجهات |
| Basis | قاعدة |
| Eigenvalue | قيمة ذاتية |
| Eigenvector | متجة ذاتي |
| Polynomial | كثيرة حدود |
| linear combination | التركيب الخطي |
| Identity matrix | مصفوفة محايدة |
| Inverse | معكوس |
| Characteristic | مميزة |
| Kernel | نواة |
| Range | المدى |
| Angle | زاوية |

# 1 Real Vector Spaces

In general, scalars for a vector space can be any field. Vector spaces with scalars from real numbers are called real vector spaces. The word "vector" means an element of a vector space.
For this course we will be concerned exclusively with real vector spaces and by scalars we mean real numbers.

**Definition 1.1.** Let $V$ be any nonempty set on which two operations are defined: addition in $V$, and multiplication by scalars. By scalar multiplication we mean a rule for associating with each scalar $a$ and each element $u \in V$ an element $a.u$. We call $V$ a vector space, if the following axioms are satisfied for all elements $u, v, w \in V$ and scalars $a$ and $b$.

1. $u + v \in V$
2. $u + (v + w) = (u + v) + w$
3. $u + v = v + u$
4. $0 \in V$ (zero vector) such that $0 + u = u + 0 = u$ for all $u \in V$.
5. For any $u \in V$, we have $-u \in V$ such that $u + (-u) = (-u) + u = 0$.
6. $a.u \in V$ for any scalar $a$ and any $u \in V$.
7. $(a + b).u = a.u + b.u$
8. $a.(u + v) = a.u + a.v$
9. $a.(b.u) = (ab).u$
10. $1.u = u$

**Example 1.1.** *Let $V = \{0\}$. Then all vector space axioms are trivially satisfied, and this simplest example of a vector space called "zero vector space."*

**Example 1.2.** *Let $V = \mathbb{R}^n$. Then it forms a vector space with usual addition and scalar multiplication of $n-$tuple.*

**Example 1.3.** *Let $V = M_{m \times n}(\mathbb{R})$, the set of all $m \times n$ matrices with real entries. Then it forms a vector space with usual addition and scalar multiplication of matrices.*

**Example 1.4.** *Let $V = \mathbb{Z}$, the set of integers. Then with respect to usual addition and scalar multiplication of numbers, it does not forms a vector space as for any non integer scalar $r$, $r \times 1 = r \notin \mathbb{Z}$.*

Now in the following theorem we study some basic properties in a vector space. Note that 0 means zero scalar and **0** means zero vector.

**Theorem 1.5.** *Let $V$ be a vector space. Then for any scalar $a$ and any vector $u \in V$.*

**(i).** $0.u = \boldsymbol{0}$

**(ii).** $a.\boldsymbol{0} = \boldsymbol{0}$

**(iii).** $(-1).u = -u$

**(iv).** *If $a.u = \boldsymbol{0}$, then either $a = 0$ or $u = \boldsymbol{0}$.*

*Proof.* Part (i). For any vector $u \in V$ we can write,

$$0.u + 0.u = (0 + 0).u = 0.u.$$

Now by axiom 4 we have $-0.u \in V$, add it both sides.

$$[0.u + 0.u] + (-0.u) = 0.u + (-0.u)$$

$$0.u + [0.u + (-0.u)] = \mathbf{0}$$

$$0.u + \mathbf{0} = \mathbf{0}$$

$$0.u = \mathbf{0}.$$

Part (ii). is exercise.
Part (iii). For any vector $u \in V$ we can write,

$$(-1).u + u = (-1).u + 1.u = (-1 + 1).u = 0.u = \mathbf{0}.$$

$$(-1).u = -u.$$

Part (iv). For any vector $u \in V$ and a scalar $a$, if $a.u = \mathbf{0}$, then either $a = 0$ or $a \neq 0$.
If $a = 0$, then we are done.
If $a \neq 0$, then we have scalar $a^{-1}$ such that,

$$a^{-1}(a.u) = a^{-1}(\mathbf{0})$$

$$(a^{-1}a).u = \mathbf{0}$$
$$1.u = \mathbf{0}$$
$$u = \mathbf{0}.$$

□

Now we see some questions related to vector spaces.

**Exercise 1.6.** *Show that the set $V = \mathbb{R}^2 = \{(x, y)|\ x, y \in \mathbb{R}\}$ forms a vector space with usual addition and scalar multiplication.*

**Solution:** Consider $u = (x_1, y_1),\ v = (x_2, y_2)$ and $w = (x_3, y_3)$ in $V = \mathbb{R}^2$ and scalars $a$ and $b$.
1. $u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in V$.
2. $u+(v+w) = (x_1, y_1)+[(x_2, y_2)+(x_3, y_3)] = (x_1+(x_2+x_3), y_1+(y_2+y_3)) = ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3) = (u + v) + w$.
3. $u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = v + u$.
4. $0 = (0, 0) \in V$ such that $0 + u = (0, 0) + (x_1, y_1) = (x_1 + 0, y_1 + 0) = (x_1, y_1) = u$, also $u + 0 = u$.
5. For any $u = (x, y) \in V$ we have $-u = (-x, -y) \in V$ such that $u+(-u) = (x, y) + (-x, -y) = (0, 0) = 0$ also $(-u) + u = 0$.
6. For any $u = (x, y) \in V$ and a scalar $a$, $a.u = a.(x, y) = (ax, ay) \in V$.
7. $(a + b).u = (a + b).(x, y) = ((a + b)x, (a + b)y) = (ax + bx, ay + by) = (ax, ay) + (bx, by) = a.(x, y) + b.(x, y) = a.u + b.u$.
8. $a.(u + v) = a.(x_1 + x_2, y_1 + y_2) = (ax_1 + ax_2, ay_1 + ay_2) = (ax_1, ay_1) + (ax_2, ay_2) = a.(x_1, y_1) + a.(x_2, y_2) = a.u + a.v$.
9. $a.(b.u) = a.[b.(x, y)] = a.(bx, by) = (a(bx), a(by)) = ((ab)x, (ab)y) = (ab).(x, y) = (ab).u$.
10. $1.u = 1.(x, y) = (1.x, 1.y) = (x, y) = u$.
Hence $V = \mathbb{R}^2$ is a vector space.

**Exercise 1.7.** *Show that the set $V = \boldsymbol{C} = \{x + yi|\ x, y \in \mathbb{R}\}$ forms a vector space with usual addition and scalar multiplication of complex numbers.*

**Solution:** Similar to Exercise 1.6 (done in class), in fact $\mathbb{R}^2$ and $\mathbb{C}$ has same additive structure.

**Exercise 1.8.** *Show that the set $V = \mathbb{R}^2 = \{(x, y)|\ x, y \in \mathbb{R}\}$ is a vector space with usual addition and given scalar multiplication.*
*For a vector $u = (x, y) \in V$ and a scalar $a$.*

**(i).** $a.(x, y) = (ay, ax)$,

**(ii).** $a.u = (ax, y)$,

**(iii).** $a.u = (ax, 0)$.

**Solution:** Part (i). Consider $u = (x, y) \in V$ such that $x \neq y$.
Now $1.(x, y) = (1 \times y, 1 \times x) = (y, x) \neq u$.
Axiom 10 fails to hold, hence $V$ is not a vector space with given scalar multiplication.
Part (ii). Consider $u = (x, y) \in V$ such that $y \neq 0$ and two scalars $a$ and $b$.
Now $(a + b).u = (a + b).(x, y) = ((a + b)x, y) = (ax + bx, y)$
but $a.u + b.u = a.(x, y) + b.(x, y) = (ax, y) + (bx, y) = (ax + bx, 2y)$. Implies $(a + b).u \neq a.u + b.u$, axiom 7 fails, hence $V$ is not a vector space with given scalar multiplication.
Part (iii). Similar to Part (i). (exercise).

**Exercise 1.9.** *Consider* $V = \mathbb{R}^2 = \{(x, y)|\ x, y \in \mathbb{R}\}$ *with following addition and scalar multiplication.*
*For a vector* $u = (x_1, y_1), v = (x_2, y_2) \in V$ *and a scalar* $a$ *we define,*

$$u + v = (x_1 + x_2 + 1, y_1 + y_2 + 1),\ a.u = (ax_1, ay_1).$$

**(i).** *If* $u = (-1, 2), v = (2, -4) \in V$, *then find* $3u + 2v$.

**(ii).** *Is* $V$ *a vector space with given addition and scalar multiplication. If no, state which axiom fails.*

**(iii).** *Show that* $0 = (-1, -1)$ *is the additive identity.*

**Solution:** Part (i).

$$3u+2v = 3(-1, 2)+2(2, -4) = (-3, 6)+(4, -8) = (-3+4+1, 6+(-8)+1) = (2, -1).$$

Part (ii). Consider any $u = (x, y) \in V$ and two scalars $a$ and $b$.
Now $(a + b).u = (a + b).(x, y) = ((a + b)x, (a + b)y) = (ax + bx, ay + by)$
but $a.u+b.u = a.(x, y)+b.(x, y) = (ax, ay)+(bx, by) = (ax+bx+1, ay+by+1)$.
Implies $(a + b).u \neq a.u + b.u$, axiom 7 fails, hence $V$ is not a vector space with given addition.
Part (iii). Consider any $u = (x, y) \in V$ and $0 = (-1, -1) \in V$.
Now $0 + u = (-1, -1) + (x, y) = (-1 + x + 1, -1 + y + 1) = (x, y) = u$.

Similarly we can see that $u + 0 = u$. Hence $0 = (-1, -1)$ is the additive identity in $V$ with given addition.

Followings are exercise for home work.

**Exercise 1.10. (a).** *Let* $V = \mathbb{R}^2 = \{(x, y)|\ x, y \geq 0\}$ *with usual addition and scalar multiplication. Is it a vector space? Justify your answer.*

**(b).** *Let* $V = \mathbb{R}^2 = \{(x, \frac{1}{2}x)|\ x \in \mathbb{R}\}$ *with usual addition and scalar multiplication. Is it a vector space?*

# 2 Subspaces

It is possible for one vector space to be contained within another. We will explore this idea in this section, we will discuss how to recognize such vector spaces, and we will give a variety of examples that will be used in our later work.

**Definition 2.1.** A subset $W$ of a vector space $V$ is called a **subspace** of $V$ if $W$ is itself a vector space under the addition and scalar multiplication defined on $V$.

**Theorem 2.1.** *If $W$ is a set of one or more vectors in a vector space $V$, then $W$ is a subspace of $V$ if and only if the following conditions hold.*

**(a).** *If $u$ and $v$ are vectors in $W$, then $u + v$ is in $W$.*

**(b).** *If $k$ is any scalar and $u$ is any vector in $W$, then $ku$ is in $W$.*

*Proof.* If $W$ is a subspace of $V$, then all the vector space axioms hold in $W$, including Axioms 1 and 6, which are precisely conditions $(a)$ and $(b)$.
Conversely, assume that conditions $(a)$ and $(b)$ hold. Since these are Axioms 1 and 6, and since Axioms 2, 3, 7, 8, 9, and 10 are inherited from $V$, we only need to show that Axioms 4 and 5 hold in $W$. For this purpose, let $u$ be any vector in $W$. It follows from condition $(b)$ that $ku$ is a vector in $W$ for every scalar $k$. In particular, $0u = 0$ and $(-1)u = -u$ are in $W$, which shows that Axioms 4 and 5 hold in $W$. □

**Remark 2.2.** *Note that every vector space has at least two subspaces, itself and its zero subspace.*

**Example 2.3.** *If $V$ is any vector space, and if $W = \{0\}$ is the subset of $V$ that consists of the zero vector only, then $W$ is closed under addition and scalar multiplication since*

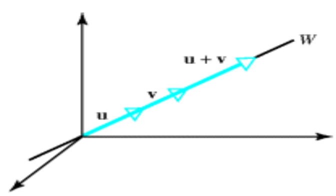
$$0 + 0 = 0$$

*and*

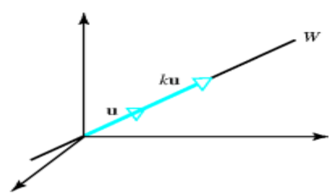
$$k0 = 0$$

*for any scalar $k$. We call $W$ the **zero subspace** of $V$.*

**Example 2.4.** *If $W$ is a line through the origin of either $R^2$ or $R^3$, then adding two vectors on the line $W$ or multiplying on the line $W$ by a scalar produces another vector on the line $W$, so $W$ is closed under addition and scalar multiplication (see the following Figure for an illustration in $R^3$).*

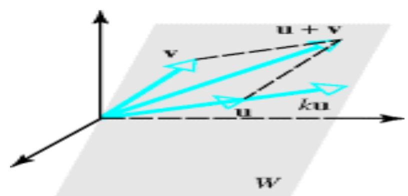



(*a*) *W* is closed under addition.



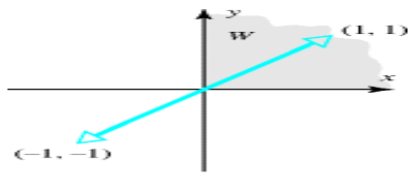

(*b*) *W* is closed under scalar multiplication.

**Example 2.5.** *If $u$ and $v$ are vectors in a plane $W$ through the origin of $R^3$, then it is evident geometrically that $u + v$ and $ku$ lie in the same plane $W$ for any scalar $k$. Thus $W$ is closed under addition and scalar multiplication.*




The vectors u + v and *k*u both lie in the same plane as u and v

**Example 2.6.** *Let $W$ be the set of all points $(x, y)$ in $R^2$ for which $x \geq 0$ and $y \geq 0$ (the shaded region in the following Figure). This set is not a subspace of $R^2$ because it is not closed under scalar multiplication. For example, $v = (1, 1)$ is a vector in $W$, but $(-1)v = (-1, -1)$ is not.*




*W* is not closed under scalar multiplication

**Example 2.7.** *We know that the sum of two symmetric $n \times n$ matrices is symmetric and that a scalar multiple of a symmetric $n \times n$ matrix is symmetric. Thus, the set of symmetric $n \times n$ matrices is closed under addition and scalar multiplication and hence is a subspace of $M_{nn}$. Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of $M_{nn}$.*

**Example 2.8.** *Recall that a* polynomial *is a function that can be expressed in the form*

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

*where $a_0,\ a_1,\ \cdots,\ a_n$ are constants. Now the sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial. Thus, the set of all polynomials $\mathbb{R}[x]$ is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by $P_\infty$.*

**Exercise 2.9.** *Decide which of the following subsets are subspaces of $\mathbb{R}^2$.*

**(a).** $W_1 = \{(a, 0) |\ a \in \mathbb{R}\}$.

**(b).** $W_2 = \{(a, b) |\ a + b = 1\}$.

**Solution:** Part **(a).** Clearly $W_1$ is non empty, since $(0, 0) \in W_1$.
Take any two elements $(a_1, 0),\ (a_2, 0) \in W_1$, where $a_1, a_2 \in W_1$.
Now $(a_1, 0) + (a_2, 0) = (a_1 + a_2, 0) \in W_1$ as $a_1 + a_2 \in \mathbb{R}$.
Also for any $(a, 0) \in W_1$ and a scalar $k$, we have $k.(a, 0) = (ka, 0) \in W_1$.
Hence $W_1$ is a subspace of $\mathbb{R}^2$.
Part **(b).** Since $(1, 0), (0, 1) \in W_2$ but $(1, 0) + (0, 1) = (1, 1) \notin W_2$ or the identity $(0, 0) \notin W_2$. Hence $W_2$ is not a subspace of $\mathbb{R}^2$.

**Exercise 2.10.** *Decide which of the following subsets are subspaces of* $\mathbb{R}^3$.

**(a).** $W_1 = \{(a, b, c) |\ c = a + b\}$.

**(b).** $W_2 = \{(a, b, 1) |\ a, b \in \mathbb{R}\}$.

**Solution:** Part **(a).** Since $(0, 0, 0) \in W_1$ gives $W_1$ is non empty.
Take any $(a_1, b_1, c_1),\ (a_2, b_2, c_2) \in W_1$, where $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$.
As $c_1 + c_2 = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$, therefore $(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2) \in W_1$.
Also for any scalar $k$ and a $(a, b, c) \in W_1$ we have $c = a + b$ or $kc = ka + kb$ which gives $k.(a, b, c) = (ka, kb, kc) \in W_1$. Hence $W_1$ is a subspace of $\mathbb{R}^3$.
Part **(b).** For any two elements $(a_1, b_1, 1),\ (a_2, b_2, 1) \in W_2$,
we have $(a_1, b_1, 1) + (a_2, b_2, 1) = (a_1 + a_2, b_1 + b_2, 2) \notin W_2$.
Hence $W_2$ is not a subspace of $\mathbb{R}^3$.

**Exercise 2.11.** *Decide which of the following subsets are subspaces of* $M_{2\times 2}(\mathbb{R})$.

**(a).** $W_1 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} |\ a, b, c \in \mathbb{R} \right\}$.

**(b).** $W_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} |\ ad - bc = 0 \right\}$.

**Solution:** Part **(a).** Clearly $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W_1$, so $W_1$ is non empty.
Take $A = \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} \in W_1$.
Now $A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ b_1 + b_2 & c_1 + c_2 \end{bmatrix} \in W_1$.

Also for any scalar $k$ and a matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in W_1$,
we have $k.A = \begin{bmatrix} ka & kb \\ kb & kc \end{bmatrix} \in W_1$. Hence $W_1$ is a subspace of $M_{2\times 2}(\mathbb{R})$.
Part **(b).** Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W_2$.
But $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W_2$. Hence $W_2$ is not a subspace of $M_{2\times 2}(\mathbb{R})$.

**Exercise 2.12.** *Decide which of the following subsets are subspaces of* $\mathbb{R}[x]$.

**(a).** $W_1 = \{a + bx |\ b \neq 0\}$.

**(b).** $W_2 = \{a + bx + cx^2 |\ a + b + c = 0\}$.

**Solution:** Part **(a).** For $f = 1 + x,\ g = 1 - x \in W_1$ we have $f + g = 2 \notin W_1$ or additive identity $0 \notin W_1$. Hence $W_1$ is not a subspace of $\mathbb{R}[x]$.
Part **(b).** Since $0 + 0x + 0x^2 = 0 \in W_2$ implies $W_1$ is non empty.
Take $f = a_1 + b_1x + c_1x^2,\ g = a_2 + b_2x + c_2x^2 \in W_2$, so $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 0$. Now

$$f + g = (a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)$$

$$= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2$$

with $(a_1+a_2)+(b_1+b_2)+(c_1+c_2) = (a_1+b_1+c_1)+(a_2+b_2+c_2) = 0+0 = 0$, which gives $f + g \in W_2$.
For any scalar $k$ and any $f = a + bx + cx^2 \in W_2$ with $a + b + c = 0$.
We have $k.f = (ka)+(kb)x+(kc)x^2$ such that $(ka)+(kb)+(kc) = k(a+b+c) = k(0) = 0$. Hence $k.f \in W_2$ which concludes $W_2$ is a subspace of $\mathbb{R}[x]$.

Followings are exercise for home work.

**Exercise 2.13.** **(a).** *Show that the set*

$$W_1 = \{(x, y, z)|\ x, y, z \in \mathbb{R} \ \textit{and}\ x \cdot x = z \cdot z\}$$

*is not a subspace of* $\mathbb{R}^2$.

**(b).** *Show that* $W_2 = \{A \in M_{2\times 2}(\mathbb{R})|\ A = A^T\}$ *is a subspace of* $M_{2\times 2}(\mathbb{R})$.

## Building Subspaces

The following theorem provides a useful way of creating a new subspace from known subspaces.

**Theorem 2.14.** *If $W_1$, $W_2$, $\cdots$, $W_r$ are subspaces of a vector space $V$, then the intersection of these subspaces is also a subspace of $V$.*

*Proof.* Let $W$ be the intersection of the subspaces $W_1$, $W_2$, $\cdots$, $W_r$. This set is not empty because each of these subspaces contains the zero vector of $V$, and hence so does their intersection. Thus, it remains to show that $W$ is closed under addition and scalar multiplication.
To prove closure under addition, let $u$ and $v$ be vectors in $W$. Since $W$ is the intersection of $W_1$, $W_2$, $\cdots$, $W_r$, it follows that $u$ and $v$ also lie in each of these subspaces. Since these subspaces are all closed under addition, they all contain the vector $u + v$ and hence so does their intersection $W$. This proves that $W$ is closed under addition. We leave the proof that $W$ is closed under scalar multiplication to you. $\square$

The union of any two subspaces is not a subspace in general as shown in the following question.

**Exercise 2.15.** *Consider two subspaces $W_1 = \{(a, 0)|\ a \in \mathbb{R}\}$ and $W_2 = \{(0, b)|\ b \in \mathbb{R}\}$ of $\mathbb{R}^2$. Show that union $W_1 \cup W_2$ is not a subspaces of $\mathbb{R}^2$.*

**Solution:** Take $(1, 0),\ (0, 1) \in W_1 \cup W_2$ but $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$. Hence $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$.

In order to achieve smallest subspace which contains any given two subspaces of a vector space, we state following.

**Theorem 2.16.** *Let $U$ and $W$ be any two subspaces of a vector spaces $V$. Then*

$$U + W = \{u + w|\ u \in U,\ w \in W\}$$

*is also a subspace of $V$, called **sum of subspaces.***

*Proof.* Since $0 \in U$ and $0 \in W$, so $0 + 0 = 0 \in U + W$ which gives $U + W$ is non empty.
For any $u_1 + w_1,\ u_2 + w_2 \in U + W$, where $u_1,\ u_2 \in U$ and $w_1,\ w_2 \in W$.
Now $(u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2) \in U + W$ as $u_1 + u_2 \in U$

and $w_1 + w_2 \in W$.
Also for any scalar $k$ and any $u + w \in U + W$, where $u \in U$ and $w \in W$.
We have $k.(u + w) = ku + kw \in U + W$ as $ku \in U$ and $kw \in W$.
Hence $U + W$ us also a subspace of $V$. □

**Definition 2.2.** A vector space $V$ is called a **direct sum** of two subspaces $U$ and $W$, if

**1.** $U + W = V$.

**2.** $U \cap W = \phi$.

We write it as, $V = U \oplus W$.

**Example 2.17.** *Consider two subspaces* $W_1 = \{(a, 0)|\ a \in \mathbb{R}\}$ *and* $W_2 = \{(0, b)|\ b \in \mathbb{R}\}$ *of* $\mathbb{R}^2$. *Then* $\mathbb{R}^2 = W_1 \oplus W_2$.

# 3 Linear Combination and Span

Sometimes we will want to find the smallest subspace of a vector space $V$ that contains all of the vectors in some set of interest. The following section will help us to do that.

**Definition 3.1.** If $w$ is a vector in a vector space $V$, then $w$ is said to be a **linear combination** of the vectors $v_1, v_2, \ldots, v_n$ in $V$ if $w$ can be expressed in the form,

$$w = a_1v_1 + a_2v_2 + a_3v_3 + \ldots + a_nv_n,$$

where $a_1, a_2, a_3, \ldots, a_n$ are scalars. These scalars are called **coefficients** of the linear combination.

**Example 3.1.** *Consider the vectors* $u = (1, 2, -1)$ *and* $v = (6, 4, 2)$ *in* $\mathbb{R}^3$*. Show that* $w = (9, 2, 7)$ *is a linear combination of* $u$ *and* $v$ *and that* $w' = (4, -1, 8)$ *is not a linear combination of* $u$ *and* $v$*.*

**Solution:** For $w$ to be a linear combination of $u$ and $v$, there must be scalars, say $a$ and $b$ such that; $w = a.u + b.v$ that is,

$$(9, 2, 7) = a(1, 2, -1) + b(6, 4, 2)$$

or

$$(9, 2, 7) = (a + 6b, 2a + 4b, -a + 2b)$$

Equating corresponding components gives,

$$a + 6b = 9$$

$$2a + 4b = 2$$

$$-a + 2b = 7.$$

Solving this system we get $a = -3$ and $b = 2$. Now $w = (-3).u + 2.v$ (verify). Similarly for $w'$ to be a linear combination of $u$ and $v$, there must be scalars, say $a$ and $b$ such that; $w' = a.u + b.v$, similarly following above steps and equation corresponding components we get,

$$a + 6b = 4$$

$$2a + 4b = -1$$

$$-a + 2b = 8.$$

There are no such scalars $a$ and $b$ exist that is above system of equations is inconsistent (verify). Consequently, $w'$ is not a linear combination of $u$ and $v$.

**Theorem 3.2.** *If $S = \{v_1, v_2, v_3, \ldots, v_n\}$ is a nonempty set of vectors in a vector space $V$, then:*

**(a).** *The set $W$ of all possible linear combinations of the vectors in $S$ is a subspace of $V$.*

**(b).** *The set $W$ in part (a) is the smallest subspace of $V$ that contains all of the vectors in $S$ in the sense that any other subspace that contains those vectors, also contains $W$.*

*Proof.* Part (a). Let $W$ be the set of all possible linear combinations of the vectors in $S$, since $0 = 0.v_1 + 0.v_2 + \ldots + 0.v_n \in W$ which shows that $W$ is non empty. Now we must show that $W$ is closed under addition and scalar multiplication. To prove closure under addition, let

$$u = a_1v_1 + a_2v_2 + \ldots + a_nv_n \text{ and } v = b_1v_1 + b_2v_2 + \ldots + b_nv_n \in W,$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are any scalars. The sum,

$$u + v = (a_1v_1 + a_2v_2 + \ldots + a_nv_n) + (b_1v_1 + b_2v_2 + \ldots + b_nv_n)$$

$$= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \ldots + (a_n + b_n)v_n \in W.$$

Now for any scalar $k$ and a vector $u = a_1v_1 + a_2v_2 + \ldots + a_nv_n \in W$ we have,

$$k.u = k.(a_1v_1 + a_2v_2 + \ldots + a_nv_n) = (ka_1)v_1 + (ka_2)v_2 + \ldots + (ka_n)v_n \in W.$$

Hence $W$ is a subspace of $V$.

Part (b). Let $W'$ be any subspace of $V$ that contains all of the vectors in $S$. Since $W'$ is closed under addition and scalar multiplication, it contains all linear combinations of the vectors in $S$ and hence contains $W$. □

**Definition 3.2.** The subspace of a vector space $V$ that is formed from all possible linear combinations of the vectors in a nonempty set $S$ is called the **span of** $S$, and we say that the vectors in $S$ span that subspace.

If $S = \{v_1, v_2, v_3, \ldots, v_n\}$, then we denote the span of $S$ by,

$$\text{span}(S) = \{a_1v_1 + a_2v_2 + \ldots + a_nv_n |\ a_1, a_2, \ldots, a_n \text{ are any scalars}\}.$$

**Example 3.3.** *The vectors* $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ *and* $e_3 = (0, 0, 1)$ *spans* $\mathbb{R}^3$. *That is, any vector* $v = (a, b, c) \in \mathbb{R}^3$ *can be written as,* $v = ae_1 + be_2 + ce_3$.

*In general, the vectors* $\{e_1 = (1, 0, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0), \ldots e_n = (0, 0, 0, \ldots, 1)\}$ *spans* $\mathbb{R}^n$.

**Example 3.4.** *The vectors* $\{1, x, x^2, \ldots, x^n\}$ *spans* $P_n(x)$ *(the set of all polynomials with degree at most* $n$*).*

# 4 Linear Independence and Dependence

In this section we will consider the question of whether the vectors in a given set are interrelated in the sense that one or more of them can be expressed as a linear combination of the others.

**Definition 4.1.** If $S = \{v_1, v_2, v_3, \ldots, v_n\}$ is a nonempty set of vectors in a vector space $V$, then the equation,

$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$$

has at least one solution, namely, $a_1 = a_2 = a_3 = \ldots = a_n = 0$.

We call this the **trivial solution**.

If this is the only solution, then $S$ is said to be a **linearly independent** set.

If there are solutions in addition to the trivial solution, then S is said to be a **linearly dependent** set.

**Example 4.1.** *The set,* $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *is linearly independent in* $\mathbb{R}^3$*, because if we consider the equation,*

$$a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = (0, 0, 0),$$

*then it has only trivial solution, that is* $a_1 = a_2 = a_3 = 0$.

*In general, the set* $S = \{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots (0, 0, 0, \ldots, 1)\}$ *is linearly independent in* $\mathbb{R}^n$.

**Example 4.2.** *The vectors* $S = \{1, x, x^2, \ldots, x^n\}$ *is also linearly independent in the vector space* $P_n(x)$.

**Exercise 4.3.** *Decide which of the following sets are linearly independent in the vector space* $\mathbb{R}^2$.

**(a).** $S_1 = \{(5, -1), (0, 3)\}$.

**(b).** $S_2 = \{(1, -2), (5, -10)\}$.

**Solution:** Part (a). Consider the equation $a(5, -1) + b(0, 3) = (0, 0)$ for any scalars $a$ and $b$. After simplification we get,

$$5a = 0 \text{ and } -a + 3b = 0$$

which implies $a = b = 0$. Hence $S_1$ is linearly independent in $\mathbb{R}^2$.
Part (b). Consider the equation $a(1, -2) + b(5, -10) = (0, 0)$ for any scalars $a$ and $b$. After simplification we get,

$$a + 5b = 0 \text{ and } -2a - 10b = 0$$

which implies

$$a = -5b \text{ and } 0.b = 0.$$

It shows that $b$ can be any real number which means we have more than one (trivial) solutions. Hence $S_1$ is linearly dependent in $\mathbb{R}^2$.

Followings are exercises for home work.

**Exercise 4.4.** *Decide which of the following sets are linearly independent in the vector space* $\mathbb{R}^3$.

**(a).** $S_1 = \{(1, 2, -1), (2, 0, 3)\}$.

**(b).** $S_2 = \{(2, 0, -2), (0, -1, -5), (4, 1, 1)\}$.

**Exercise 4.5.** *Decide which of the following sets are linearly independent in the vector space* $P_2(x)$.

**(a).** $S_1 = \{3 - x + 2x^2, 1 - x^2, 2x + 5x^2\}$.

**(b).** $S_2 = \{2 + x, 1 - x^2, 1 + x + x^2\}$.

# 5 Basis and Dimension

We usually think of a line as being one-dimensional, a plane as two-dimensional, and the space around us as three dimensional. It is the primary goal of this section to make this intuitive notion of dimension precise.

**Definition 5.1.** Let $S$ be subset of a vector space $V$, then $S$ is called a basis for $V$ if the following two conditions hold:

**(a).** $S$ is linearly independent.

**(b).** $S$ spans $V$.

**Example 5.1.** *The set,* $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ *forms a basis for vector pace* $\mathbb{R}^3$*, called* ***standard basis****.*

*In general, the set* $S = \{(1,0,0,\dots,0),(0,1,0,\dots,0),\dots(0,0,0,\dots,1)\}$ *forms* ***standard basis*** *for vector space* $\mathbb{R}^n$*.*

**Example 5.2.** *The set of vectors* $S = \{1, x, x^2, \dots, x^n\}$ *forms* ***standard basis*** *for the vector space* $P_n(x)$*.*

**Example 5.3.** *The vectors* $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ *forms* ***standard basis*** *for the vector space* $M_{2\times 2}$*.*

**Example 5.4.** *The basis for zero vector space* $V = \{0\}$ *is the set* $\phi$*.*

A vector space that can be spanned by finitely number of vectors is called **finite-dimensional** vector space. A vector space that cannot be spanned by finitely many vectors is said to be **infinite-dimensional** vector space.

**Example 5.5.** *The spanning set for the vector spaces* $\mathbb{R}^\infty$ *and the set of polynomials* $\mathbb{R}[x]$ *is not finite.*

Our goal now in this section is to establish basic theorems to find a basis for a vector space.

**Theorem 5.6.** *Let $V$ be a finite-dimensional vector space, and let set $S = \{v_1, v_2, v_3, \ldots, v_n\}$ be any basis for $V$.*

**(a).** *If a set has more than n vectors, then it is linearly dependent.*

**(b).** *If a set has less than n vectors, then it does not span V.*

For Proof follow book [1]. Above Theorem 5.6 enable us to state following fundamental result.

**Theorem 5.7.** *All bases for a finite-dimensional vector space have the same number of vectors.*

As the number of vectors in a basis for a vector space is constant, this leads to define dimension.

**Definition 5.2.** The dimension of a finite-dimensional vector space $V$ is the number of vectors in a basis for $V$. We denote it as, $\dim(V)$.

The zero vector space is defined to have dimension zero.

**Example 5.8.**

$$\dim(\mathbb{R}^n) = n$$

$$\dim(P_n(x)) = n + 1$$

$$\dim(M_{m\times n}(\mathbb{R})) = m \times n.$$

**Example 5.9.** *(Infinite dimensional vector spaces)*

*The vector space $\mathbb{R}^\infty$ and the vector space $\mathbb{R}[x]$ are infinite dimensional.*

The following theorem is called plus minus theorem.

**Theorem 5.10.** *Let $S$ be a nonempty set of vectors in a vector space V.*

**(a).** *If $S$ is a linearly independent set, and if $v$ is a vector in $V$ that is outside of span$(S)$, then the set $S \cup \{v\}$ is still linearly independent.*

**(b).** *If $v$ is a vector in $S$ that is expressible as a linear combination of other vectors in $S$, then the set span$(S)$=span$(S - \{v\})$.*

In general, to show that a set of vectors $\{v_1, v_2, v_3, \dots, v_n\}$ is a basis for a vector space $V$, we must show that the vectors are linearly independent and span $V$. However, if we happen to know that $V$ has dimension $n$, then it suffices to check either linear independence or spanning the remaining condition will hold automatically. This is the content of the following theorem.

**Theorem 5.11.** *Let $V$ be an $n$-dimensional vector space, and let $S = \{v_1, v_2, v_3, \dots, v_n\} \subseteq V$. Then $S$ is a basis for $V$ if and only if $S$ is linearly independent or $S$ spans $V$.*

The next theorem reveals two important facts about the vectors in a finite-dimensional vector space $V$ :
1. Every spanning set for a subspace is either a basis for that subspace or has a basis as a subset.
2. Every linearly independent set in a subspace is either a basis for that subspace or can be extended to a basis for it.

**Theorem 5.12.** *Let $S$ be a finite set of vectors in a finite-dimensional vector space V.*

**(a).** *If $S$ spans $V$ but is not a basis for V, then $S$ can be reduced to a basis for $V$ by removing appropriate vectors from $S$.*

**(b).** *If $S$ is a linearly independent set that is not a basis for $V$, then $S$ can be enlarged to a basis for $V$ by inserting appropriate vectors into $S$.*

For the proofs of above theorems please read book [1].

**Exercise 5.13.** *Decide which of the following set of vectors forms a basis for corresponding vector spaces.*

**(i).** $S_1 = \{(1, -2), (-1, 3)\} \subset \mathbb{R}^2$.

**(ii).** $S_2 = \{(1, 2), (2, 3), (2, 4)\} \subset \mathbb{R}^2$.

**(iii).** $S_3 = \{(1, 2, 3), (4, 5, 6)\} \subset \mathbb{R}^3$.

**(iv).** $S_4 = \{(1, 2, 1), (3, -2, 0), (-1, 1, 1)\} \subset \mathbb{R}^3$.

**(v).** $S_5 = \{1 - x, 2x + x^2, 1 - 3x - x^2\} \subset P_2(x)$.

**(vi).** $S_6 = \left\{ \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right\} \subset M_{2\times2}(\mathbb{R})$.

**Solution:** Part (i). As $|S_1| = 2 = \dim(\mathbb{R}^2)$. We first see that $S_1$ is linearly independent? Take $a(1, -2) + b(-1, 3) = (0, 0)$ for any scalars $a$ and $b$. Implies $(a - b, -2a + 3b) = (0, 0)$, which gives $a - b = 0$ and $-2a + 3b = 0$, or $a = b$ using in the second equation, we get $-2a + 3a = 0 \Rightarrow a = 0$ and hence $b = 0$. Which shows that $S_1$ is linearly independent and by Theorem 5.11, forms a basis for $\mathbb{R}^2$.

**Note:** In the case where number of vectors in a set is equal to the dimension of the space $\mathbb{R}^n$ or $P_n(x)$, we can verify linearly independence of sets by taking determinant of matrix whose rows are entries in the vectors. For example in above part we can take, $\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} = 1 \neq 0$. Hence $S_1$ is linearly independent and rest follows as above.

Part (ii). Since $|S_2| = 3 > \dim(\mathbb{R}^2) = 2$. By Theorem 5.6, it follows that $S_2$ is linearly dependent and hence does not forms a basis for $\mathbb{R}^2$.
Part (iii). Since $|S_3| = 2 < \dim(\mathbb{R}^3) = 3$. By Theorem 5.6, it follows that $S_3$ does not span $\mathbb{R}^3$ and hence does not forms a basis for $\mathbb{R}^3$.
Part (iv). Since $|S_4| = 3 = \dim(\mathbb{R}^3) = 3$, so we see if $S_4$ is linearly independent?
Take $\begin{vmatrix} 1 & 2 & 1 \\ 3 & -2 & 0 \\ -1 & 1 & 1 \end{vmatrix} = 1(-2-0) - 2(3-0) + 1(3-2) = -5 \neq 0.$
Which shows that $S_4$ is linearly independent and by Theorem 5.11, forms a basis for $\mathbb{R}^3$.
Part (v). Since $|S_5| = 3 = \dim(P_2(x)) = 3$, so we see if $S_5$ is linearly independent?
We take coefficients of polynomials in $S_5$ and get,

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & -3 & -1 \end{vmatrix} = 1(-2+3) + 1(0-1) + 0(0-2) = 1 - 1 + 0 = 0.$$

Which shows that $S_5$ is linearly dependent and hence does not forms a basis for $P_2(x)$.
Part (vi). Since $|S_6| = 4 = \dim(M_{2\times 2}(\mathbb{R})) = 4$, so we see if $S_6$ is linearly independent?
For any scalars $a, b, c$ and $d$ we take,

$$a\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a-b+d & -a+b+c \\ a-c+d & b+c-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Implies $a-b+d=0,\ -a+b+c=0,\ a-c+d=0$ and $b+c-d=0$.
After simplification we get, $a=b=c=d=0$, which shows that $S_6$ is linearly independent and hence by Theorem 5.11 it forms a basis for $M_{2\times 2}(\mathbb{R})$.

## 5.1 Basis and Dimension of Subspaces

We conclude this section with a theorem that relates the dimension of a vector space to the dimensions of its subspaces.

**Theorem 5.14.** *If $W$ is a subspace of a finite-dimensional vector space $V$, then:*

**(a).** *$W$ is also finite-dimensional.*

**(b).** $\dim(W) \leq \dim(V)$.

**(c).** *$W = V$ if and only if* $\dim(W) = \dim(V)$.

For the proof of above theorem please read book [1].

**Exercise 5.15.** *Find a basis and dimension for the following subspaces of corresponding vector spaces.*

**(i).** $W_1 = \{(a, b) |\ a + b = 0\} \subseteq \mathbb{R}^2$.

**(ii).** $W_2 = \{(a, b, c) |\ a + 2b = c\} \subseteq \mathbb{R}^3$.

**(iii).** $W_3 = \{a + bx + cx^2 |\ b = c\} \subseteq P_2(x)$.

**(iv).** $W_4 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} |\ a + d = b + c \right\} \subseteq M_{2\times2}(\mathbb{R})$.

**Solution:** Part (i). For any vector $(a, b) \in W_1$ we have $a + b = 0 \Rightarrow a = -b$, which gives,

$$(a, b) = (-b, b) = b(-1, 1),$$

implies $W_1 =$span$\{(-1, 1)\}$ also $\{(-1, 1)\}$ is linearly independent and hence it forms a basis for $W_1$. Clearly $\dim(W_1) = 1$.

Part (ii). For any vector $(a, b, c) \in W_2$ we have $a + 2b = c$, which gives,

$$(a, b, c) = (a, b, a + 2b) = (a, 0, a) + (0, b, 2b) = a(1, 0, 1) + b(0, 1, 2),$$

implies $W_2$ =span$\{(1,0,1),(0,1,2)\}$ also it is easy to verify that the set $\{(1,0,1),(0,1,2)\}$ is linearly independent and hence forms a basis for $W_2$. Clearly $\dim(W_2)=2$.
Part (iii). For any vector $a+bx+cx^2 \in W_3$ we have $b=c$, which gives,

$$a+bx+cx^2 = a+bx+bx^2 = (a)+(bx+bx^2) = a(1)+b(x+x^2),$$

implies $W_3$ =span$\{1, x+x^2\}$ also it is easy to verify that $\{1, x+x^2\}$ is linearly independent and hence it forms a basis for $W_3$. Clearly $\dim(W_3)=2$.
Part (iv). For any vector $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_4$ we have $a+d=b+c \Rightarrow a=b+c-d$, which gives,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b+c-d & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} -d & 0 \\ 0 & d \end{bmatrix}$$

$$= b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

implies $W_4$ =span$\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ also it is easy to verify that it is linearly independent and hence forms a basis for $W_4$. Clearly $\dim(W_4)=3$.

# 6 Row Space, Column Space and Null Space

In this section we will study some important vector spaces that are associated with matrices. Our work here will provide us with a deeper understanding of the relationships between the solutions of a linear system and properties of its coefficient matrix.
For an $m \times n$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The vectors,

$$R_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$$
$$R_2 = \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}$$
$$\vdots$$
$$R_m = \begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

in $\mathbb{R}^n$ are called **row vectors** of the matrix $A$.
The vectors,

$$C_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, C_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots C_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in $\mathbb{R}^m$ are called **column vectors** of the matrix $A$.

**Example 6.1.** *Let*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}.$$

*Then row vectors of A are,* $R_1 = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$, *and* $R_2 = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix}$.

*The column vectors of A are,* $C_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $C_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ *and* $C_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

**Definition 6.1.** If $A$ is an $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of $A$ is called the row space of $A$, and the subspace of $\mathbb{R}^m$ spanned by the column vectors of $A$ is called the column space of $A$. The solution space of the homogeneous system of equations $AX = 0$, which is a subspace of $\mathbb{R}^n$, is called the null space of $A$.

## 6.1 Basis for row space, column space and null space

We first developed elementary row operations for the purpose of solving linear systems or to find the basis for row and column spaces, and performing an elementary row operation on an augmented matrix does not change the solution set of the corresponding linear system. It follows that applying an elementary row operation to a matrix $A$ does not change the solution set of the corresponding linear system $AX = 0$, or, stated another way, it does not change the null space of $A$.
The following theorem makes it possible to find bases for the row and column spaces of a matrix in row echelon form by inspection.

**Theorem 6.2.** *If a matrix $R$ is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of $R$, and the column vectors with the leading 1's of the row vectors form a basis for the column space of $R$.*

Proof is omitted.
Since for any matrix $A$, the elementary row operations do not change the row space of a matrix, so we can find a basis for the row space of $A$ by finding a basis for the row space of any row echelon form of $A$.

The problem of finding a basis for the column space of a matrix $A$ is complicated by the fact that an elementary row operation can alter its column space. However, the good news is that *elementary row operations do not alter dependence relationships among the column vectors.* The fact that elementary row operations are reversible that they also preserve linear independence among column vectors will help the cause, as it is summarized in the following theorem.

**Theorem 6.3.** *If $A$ and $B$ are row equivalent matrices, then:*

**(a).** *A given set of column vectors of $A$ is linearly independent if and only if the corresponding column vectors of $B$ are linearly independent.*

**(b).** *A given set of column vectors of $A$ forms a basis for the column space of $A$ if and only if the corresponding column vectors of $B$ form a basis for the column space of $B$.*

Proof is omitted.
Following exercise will elaborate the whole procedure to find the basis for row space, column space and null space.

**Exercise 6.4.** *Find a basis and dimension of row space, column space and null space for the following matrices.*

**(i).** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$

**(ii).** $B = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & -1 & 1 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix}.$

**(iii).** $C = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -3 & -6 \\ 2 & 3 & -2 \\ 1 & 1 & -2 \end{bmatrix}$.

**Solution:** Part (i). We first reduce given matrix $A$ to echelon form.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}_{R_2-4R_1,\ R_3-7R_1}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix}_{\frac{-1}{3}R_2} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}_{R_3+6R_2} = R.$$

The matrix $R$ is the reduced echelon form of matrix $A$.
Hence the set $\{(1,2,3),(0,1,2)\}$ forms a basis for row space of $A$ and the set $\{(1,4,7),(2,5,8)\}$ forms a basis for column space of $A$.
Clearly dim(row space of $A$) $= 2$ and dim(column space of $A$) $= 2$.
For null space we consider $RX = 0$ instead of $AX = 0$ as both systems leads to same solution space (null space).

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_3 = a \in \mathbb{R} \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = a \\ x_2 = -2a \\ x_3 = a \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ -2a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

It shows that the set $\{(1,-2,1)\}$ forms a basis for null space of $A$ and dim(null space of $A$) $= 1$.

Part (ii). We first reduce given matrix $B$ to echelon form.

$$B = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & -1 & 1 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -3 & 2 \\ 3 & -1 & 1 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix}_{R_1+R_3}$$

$$\sim \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & -10 & 10 & -5 \\ 0 & 5 & -5 & 4 \end{bmatrix}_{R_2-3R_1,\ R_3+R_1} \sim \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 5 & -5 & 4 \end{bmatrix}_{\frac{-1}{10}R_2}$$

$$\sim \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}_{R_3-5R_3} \sim \begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\frac{2}{3}R_3} = R.$$

The matrix $R$ is the reduced echelon form of matrix $B$.
Hence the set $\{(1,3,-3,2),(0,1,-1,\frac{1}{2}),(0,0,0,1)\}$ forms a basis for row space of $B$ and the set $\{(2,3,-1),(1,-1,2),(0,1,2)\}$forms a basis for column space of $A$.
Clearly dim(row space of $B$) $= 3$ and dim(column space of $B$) $= 3$.
For null space we consider $RX = 0$.

$$\begin{bmatrix} 1 & 3 & -3 & 2 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + 3x_2 - 3x_3 + 2x_4 = 0 \\ x_2 - x_3 - \frac{1}{2}x_4 = 0 \\ x_3 = a \in \mathbb{R} \\ x_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = a \\ x_3 = a \\ x_4 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ a \\ 0 \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

It shows that the set $\{(0, 1, 1, 0)\}$ forms a basis for null space of $B$ and dim(null space of $B$) $= 1$.
Part (iii). Left as an exercise.

## 6.2 Basis and dimension for Solution Space

Now basis for the solution space of any homogeneous system of linear equations can be achieved by finding the basis for the null space of corresponding augmented matrix of the system, as shown in the following exercise.

**Exercise 6.5.** *Find a basis and the dimension for the solution space of following homogeneous systems of linear equations.*

**(i).**
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

**(ii).**
$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

**(iii).**
$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

**Solution:** Part (i). Let the augment matrix for the given system is,

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

We first reduce matrix $A$ to echelon form.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

For basis of solution space (null space) we consider $RX = 0$ instead of $AX = 0$ as both systems leads to same solution space.

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_2 = 0 \\ x_3 = a \in \mathbb{R} \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = a \\ x_2 = 0 \\ x_3 = a \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

It shows that the set $\{(1, 0, 1)\}$ forms a basis for solution space of given system of linear equations and $\dim(\text{solution space}) = 1$.

Part (ii). Let the augment matrix for the given system is,

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix}$$

We first reduce matrix $A$ to echelon form.

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & 0 & 2 \\ 5 & -1 & 1 & -1 \end{bmatrix}_{R_1 - R_2}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & -1 \\ 5 & -1 & 1 & -1 \end{bmatrix}_{\frac{-1}{2}R_1} \sim \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 4 & 1 & 4 \end{bmatrix}_{R_2 - 5R_1}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & \frac{1}{4} & 1 \end{bmatrix}_{\frac{1}{4}R_2} = R.$$

For basis of solution space (null space) we consider $RX = 0$ instead of $AX = 0$ as both systems leads to same solution space.

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 - x_2 - x_4 = 0 \\ x_2 + \frac{1}{4}x_3 + x_4 = 0 \\ x_3 = a \in \mathbb{R} \\ x_4 = b \in \mathbb{R} \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -\frac{1}{4}a \\ x_2 = -\frac{1}{4}a - b \\ x_3 = a \\ x_4 = b \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}a \\ -\frac{1}{4}a - b \\ a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}a \\ -\frac{1}{4}a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

It shows that the set $\{(-\frac{1}{4}, -\frac{1}{4}, 1, 0), (0, -1, 0, 1)\}$ forms a basis for solution space of given system of linear equations and dim(solution space) = 2.

# 7 Linear Transformations

Recall that a **function** is a rule that associates with each element of a set $A$ one and only one element in a set $B$. If $f$ associates the element $b$ with the element $a$, then we write $f(a) = b$ and we say that $b$ is the image of $a$ under $f$ or value of $f$ at $a$. The set $A$ is called the **domain** of $f$ and the set $B$ the **codomain** of $f$. The subset of the codomain that consists of all images of points in the domain is called the **range** of $f$.
For a common function the domain and codomain are simply sets, but in this text we will be concerned with functions for which the domain and codomain are vector spaces.

**Definition 7.1.** If $T : V \to W$ is a function from a vector space $V$ to a vector space $W$, then $T$ is called a **linear transformation** from $V$ to $W$ if the following two properties hold for all vectors $u, v \in V$ and any scalar $a$.

**(i).** $T(a.u) = a.T(u)$

**(ii).** $T(u + v) = T(u) + T(v)$

If $V = W$, then the linear transformation $T$ is called a **linear operator** on the vector space $V$.

**Example 7.1.** *Let $V$ and $W$ be any two vector spaces and $T : V \to W$ be a mapping such that $T(0) = 0$ for every $v \in V$. Then it is easy to see that $T$ is a linear transformation called the* ***zero transformation****.*

**Example 7.2.** *Let $V$ be any vector space. The mapping $T : V \to V$ defined by $T(v) = v$ is a linear transformation, called the* ***identity operator*** *on $V$.*

**Example 7.3.** *If $V$ is a vector space and $k$ is any scalar, then the mapping $T : V \to V$ given by $T(u) = k.u$ is a linear operator on $V$, for if $a$ is any* *scalar and if* $u$ *and* $v$ *are any vectors in* $V$*, then*

$$T(u+v) = k.(u+v) = k.u + k.v = T(u) + T(v)$$

*and*

$$T(a.u) = k.(a.u) = (ka).u = a.(k.u) = a.T(u).$$

*If* $k < 1$*, then* $T$ *is called the* ***contraction*** *of* $V$ *with factor* $k$*, and if* $k > 1$*, then it is called the* ***dilation*** *of* $V$ *with factor* $k$*.*

The following theorem describes the basic properties of vector spaces.

**Theorem 7.4.** *If* $T : V \to W$ *is a linear transformation, then*

**(a).** $T(0) = 0$.

**(b).** $T(u - v) = T(u) - T(v)$*, for any* $u, v \in V$*.*

**(c).** $T(a_1u_1 + a_2u_2 + \ldots + a_nu_n) = a_1T(u_1) + a_2T(u_2) + \ldots + a_nT(u_n)$,

*for any scalars* $a_1, a_2, \ldots a_n$ *and vectors* $u_1, u_2 \ldots, u_n$*.*

*Proof.* Part (a). Since $T(0+0) = T(0) \Rightarrow T(0)+T(0) = 0+T(0) \Rightarrow T(0) = 0$.

Part(b). For any two vectors $u$ and $v$ in $V$ we have,

$$T(u-v) = T(u+(-1)v) = T(u)+T((-1)v) = T(u)+(-1)T(v) = T(u)-T(v).$$

Part (c). For any scalars $a_1, a_2, \ldots a_n$ and vectors $u_1, u_2 \ldots, u_n$.

$$\begin{aligned} T(a_1u_1 + a_2u_2 + \ldots + a_nu_n) &= T\left((a_1u_1) + (a_2u_2 + \ldots + a_nu_n)\right) \\ &= T(a_1u_1) + T(a_2u_2 + \ldots + a_nu_n) \\ &\;\;\vdots \\ &= T(a_1u_1) + T(a_2u_2) + \ldots + T(a_nu_n) \\ &= a_1T(u_1) + a_2T(u_2) + \ldots + a_nT(u_n). \end{aligned}$$

□

**Exercise 7.5.** *Decide which of the following functions are linear transformation?*

**(i).** $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined as,* $T(x, y) = (-y, x)$.

**(ii).** $T : \mathbb{R}^3 \to \mathbb{R}^2$ *defined as,* $T(x, y, z) = (x + z, y + z)$.

**(iii).** $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined as,* $T(x, y) = (x + 1, y)$.

**(iv).** $T : M_{m\times n}(\mathbb{R}) \to M_{n\times m}(\mathbb{R})$ *defined as,* $T(A) = A^T$ *(transpose of* $A$*).*

**(v).** $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ *defined as,* $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

**Solution:** Part (i). For any vector $u = (x, y) \in \mathbb{R}^2$ and any scalar $a$,

$$T(a.u) = T(ax, ay) = (-ay, ax) = a(-y, x) = aT(x, y) = aT(u).$$

Also for any two vectors $u = (x_1, y_1), v = (x_2, y_2) \in \mathbb{R}^2$,

$$T(u + v) = T(x_1 + x_2, y_1 + y_2) = (-y_1 - y_2, x_1 + x_2) = (-y_1, x_1) + (-y_2, x_2)$$

$$= T(x_1, y_1) + T(x_2, y_2) = T(u) + T(v).$$

Hence $T$ is a linear transformation.

Part (ii). For any vector $u = (x, y, z) \in \mathbb{R}^3$ and any scalar $a$,

$T(a.u) = T(ax, ay, az) = (ax+az, ay+az) = a(x+z, y+z) = aT(x, y, z) = aT(u)$.

Also for any two vectors $u = (x_1, y_1, z_1), v = (x_2, y_2, z_2) \in \mathbb{R}^3$,

$T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2) = ((x_1 + x_2) + (z_1 + z_2), (y_1 + y_2) + (z_1 + z_2))$

$= (x_1+z_1, y_1+z_1)+(x_2+z_2, y_2+z_2) = T(x_1, y_1, z_1)+T(x_2, y_2, z_2) = T(u)+T(v)$.

Hence $T$ is a linear transformation.

Part (iii). For any vector $u = (x, y) \in \mathbb{R}^2$ and any scalar $a \neq 1$,

$$T(a.u) = T(ax, ay) = (ax + 1, ay)$$

but

$$a.T(u) = aT(x, y) = a(x + 1, y) = (ax + a, ay)$$

$\Rightarrow T(a.u) \neq a.T(u)$. Hence $T$ is not a linear transformation.

Part (iv). We use some elementary properties of transpose of matrices to show that $T$ is a linear transformation.

For any matrix $A \in M_{m\times n}(\mathbb{R})$ and any scalar $a$,

$$T(a.A) = (a.A)^T = a.A^T = a.T(A).$$

Also for any two matrices $A, B \in M_{m\times n}(\mathbb{R})$,

$$T(A + B) = (A + B)^T = A^T + B^T = T(A) + T(B).$$

Part (v). For the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ and any scalar $a = 2$,

$$T(a.A) = T(\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}) = (2)(8) - (4)(6) = -8$$

but

$$a.T(A) = 2T(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = 2(-2) = -4.$$

$\Rightarrow T(a.u) \neq a.T(u)$. Hence $T$ is not a linear transformation.

## 7.1 Kernel and Range

**Definition 7.2.** If $T : V \to W$ is a linear transformation, then the set of vectors in $V$ which $T$ maps on $0 \in W$ is called **kernel** of $T$ and is denoted by, $\ker(T)$. That is,

$$\ker(T) = \{u \in V |\ T(u) = 0\}.$$

The set of all vectors in $W$ which are images under $T$ is called **range** (or image) of $T$ and is denoted by $R(T)$ or $\text{Im}(T)$. That is,

$$R(T) = \{T(u) |\ u \in V\}.$$

**Example 7.6.** *Let $T : V \to W$ be zero linear transformation. Since $T$ maps every element of $V$ into $0 \in W$, it follows that $\ker(T) = V$ and $R(T) = \{0\}$.*

**Example 7.7.** *Let $T : V \to V$ be a linear operator. Then $T(u) = u$ for every $u \in V$, hence $\ker(T) = \{0\}$ and $R(T) = V$.*

In above examples one can see that all $\ker(T)$ and $R(T)$ are subspaces. This is a consequence of the following general theorem.

**Theorem 7.8.** *If $T : V \to W$ is a linear transformation, then;*

**(a).** $\ker(T)$ *is a subspace of $V$.*

**(b).** $R(T)$ *is a subspace of $W$.*

*Proof.* Part (a). Since $T(0) = 0 \Rightarrow 0 \in \ker(T)$, hence $\ker(T)$ is nonempty. For any two vectors $u, v \in \ker(T)$ and a scalar $a$.

$$T(u + v) = T(u) + T(v) = 0 + 0 = 0 \Rightarrow u + v \in \ker(T).$$

Also,

$$T(a.u) = a.T(u) = a(0) = 0 \Rightarrow a.u \in \ker(T).$$

It shows that $\ker(T)$ is a subspace of $V$.

Part (b). Again $T(0) = 0 \in R(T)$, which shows that $R(T)$ is nonempty as it contains at least zero element.

For any two elements $w_1, w_2 \in R(T)$ we have $u, v \in V$ such that $T(u) = w_1$ and $T(v) = w_2$.

Now $w_1 + w_2 = T(u) + T(v) = T(u + v) \in R(T)$, also for any scalar $a$, $a.w_1 = a.T(u) = T(a.u) \in R(T)$. Hence $R(T)$ is a subspace of $W$. $\square$

**Definition 7.3.** Let $T : V \to W$ be a linear transformation. If the range of $T$ is finite-dimensional, then its dimension is called the **rank** of $T$ denoted by $rank(T)$ and if the kernel of $T$ is finite-dimensional, then its dimension is called the nullity of T denoted by $nullity(T)$.

The proof of following theorem is optional and can be seen in [1].

**Theorem 7.9.** *If $T : V \to W$ is a linear transformation from an $n$-dimensional vector space $V$ $(n = dim(V))$ to a vector space $W$, then*

$$rank(T) + nullity(T) = n.$$

**Exercise 7.10.** *Let $T : M_{2\times2}(\mathbb{R}) \to \mathbb{R}^2$ defined by,*

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$$

*is linear. Describe its kernel and range and give the dimension of each.*

**Solution:** It should be clear that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if and only if $b = -a$ and $d = -c$. The kernel of $T$ is therefore all matrices of the form

$$\begin{bmatrix} a & -a \\ c & -c \end{bmatrix} = a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

The two matrices $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ are not scalar multiples of each other, so they must be linearly independent. Therefore the dimension of $Ker(T)$ is two, ($nullity(T) = 2$).

Now suppose that we have any vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Clearly $T\left(\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$, so the range of $T$ is all of $\mathbb{R}^2$. Thus the dimension of $R(T)$ is two, ($rank(T) = 2$).

# 8 Eigenvalues and eigenvectors

In this chapter we will focus on classes of scalars and vectors known as *eigenvalues* and *eigenvectors,* terms derived from the German word eigen, meaning *own.* In the early 1900s it was applied to matrices and matrix transformations, and today it has applications in such diverse fields as computer graphics, mechanical vibrations, heat flow, population dynamics, quantum mechanics, and economic.

First we will define the notions of eigenvalue and eigenvector and discuss some of their basic properties.

**Definition 8.1.** If $A$ is an $n \times n$ matrix, then a nonzero vector $X \in \mathbb{R}^n$ is called an **eigenvector** of $A$, if $AX$ is a scalar multiple of $X$; that is,

$$AX = \lambda X$$

for some scalar $\lambda$. The scalar $\lambda$ is called an **eigenvalue** of $A$, and $X$ is said to be an eigenvector corresponding to $\lambda$.

**Example 8.1.** *For the matrix* $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, *the vector* $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *is an eigenvector of* $A$ *and* $\lambda = 3$ *is an eigenvalue of* $A$, *since,*

$$AX = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda X.$$

## 8.1 Computing Eigenvalues and Eigenvectors

To obtain a general procedure for finding eigenvalues and eigenvectors of an $n \times n$ matrix $A$, note that the equation $AX = \lambda X$ can be rewritten as,

$$(\lambda I X - AX) = (\lambda I - A)X = 0,$$

where $I$ is the $n \times n$ identity matrix.
For a nonzero eigenvector $X$, the above equation gives,

$$\det(\lambda I - A) = 0. \tag{8.1}$$

This is called the **characteristic equation** of $A$.

**Exercise 8.2.** *Find the eigenvalues and eigenvectors for all following matrices,*

**(i).** $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

**(ii).** $B = \begin{bmatrix} 1 & 5 \\ 3 & -1 \end{bmatrix}$.

**(iii).** $C = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$.

**Solution:** Part (i). Consider the characteristic equation, $\det(\lambda I - A) = 0$.

$$\Rightarrow \det\left(\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3 & 0\\ 8 & -1\end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix}\lambda-3 & 0\\ -8 & \lambda+1\end{vmatrix} = 0$$

$$\Rightarrow (\lambda-3)(\lambda+1) = 0 \Rightarrow\ \lambda = 3 \text{ or } \lambda = -1,$$

are the required eigenvalues of $A$.
For every $\lambda$ we find its own vector(s).
Case 1: $\lambda = 1$.
Here the matrix

$$-1I_2 - A = \begin{bmatrix}-4 & 0\\ -8 & 0\end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue1 are the **non-zero** solutions to the equation $x_1 = 0$. The solutions to the equation are the vectors of the form

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

Case 2: $\lambda = 3$.
Here the matrix

$$3I_2 - A = \begin{bmatrix}0 & 0\\ -8 & 4\end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix}1 & -\frac{1}{2}\\ 0 & 0\end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue 3 are the **non-zero** solutions to the equation $x_1 - \frac{1}{2}x_2 = 0$. The solutions to the equation are the vectors of the form

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}t\\ t\end{bmatrix} = t\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}.$$

Part (ii). Consider the characteristic equation, $\det(\lambda I - B) = 0$.

$$\Rightarrow \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 5 \\ 3 & -1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 1 & -5 \\ -3 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)(\lambda + 1) - 15 = 0 \Rightarrow \lambda^2 - 16 = 0 \Rightarrow \lambda = \pm 4,$$

are the required eigenvalues of $B$.
For every $\lambda$ we find its own vector(s).
Case 1: $\lambda = -4$.
Here the matrix

$$-4I_2 - A = \begin{bmatrix} -5 & -5 \\ -3 & -3 \end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue4 are the **non-zero** solutions to the equation $x_1 + x_2 = 0$. The solutions to the equation are the vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Case 2: $\lambda = 4$.
Here the matrix

$$4I_2 - A = \begin{bmatrix} 3 & -5 \\ -3 & 5 \end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix} 1 & -\frac{5}{3} \\ 0 & 0 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue 3 are the **non-zero** solutions to the equation $x_1 - \frac{5}{3}x_2 = 0$. The solutions to the equation are the vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}.$$

Part (iii). Home work.

**Exercise 8.3.** *Find the eigenvalues and eigenvectors for all following matrices,*

**(i).** $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

**(ii).** $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

**Solution:** Part (i). Consider the characteristic equation, $\det(\lambda I - A) = 0$.

$$\Rightarrow \det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda - 3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda - 15\lambda - 8 = 0 \Rightarrow (\lambda+1)(\lambda^2 - 7\lambda - 8) = 0 \Rightarrow (\lambda+1)(\lambda+1)(\lambda-8) = 0$$

$$\Rightarrow \ \lambda = -1(\text{twice}) \text{ or } \lambda = 8,$$

are the required eigenvalues of $A$.
For every $\lambda$ we find its own vector(s).
Case 1: $\lambda = -1$.

Here the matrix

$$-1I_3 - A = \begin{bmatrix} -4 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -4 \end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue $-1$ are the **non-zero** solutions to the equation $x_1 + \dfrac{1}{2}x_2 + x_3 = 0$.
The solutions to the equation are the vectors of the form

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t - s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Case 2: $\lambda = 8$.
Here the matrix

$$8I_3 - A = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\dfrac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue 8 are the **non-zero** solutions to the equations

$$x_1 - x_3 = 0$$
$$x_2 - \frac{1}{2}x_3 = 0.$$

The solutions to the equation are the vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ \dfrac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \dfrac{1}{2} \\ 1 \end{bmatrix}.$$

Part (ii). Home work.

# 9 Inner Product Spaces

In this chapter we shall consider the topic of inner product spaces. These are vector spaces endowed with an inner product (essentially a generalisation of the dot product of vectors in $\mathbb{R}^3$) and are extremely important.

**Definition 9.1.** An **inner product** on a real vector space $V$ is a function that associates a real number $\langle u, v\rangle$ with each pair of vectors $u$ and $v$ in $V$ in such a way that the following axioms are satisfied for all vectors $u$, $v$ and $z$ in $V$ and all scalars $k$.

**(i).** $\langle u, v\rangle = \langle v, u\rangle$.

**(ii).** $\langle u + v, z\rangle = \langle u, z\rangle + \langle v, z\rangle$.

**(iii).** $\langle ku, v\rangle = k\langle u, v\rangle$.

**(iv).** $\langle v, v\rangle \geq 0$.

**(v).** $\langle v, v\rangle = 0$ if and only if $v = \mathbf{0}$.

A real vector space with an inner product is called a **real inner product space**.

**Example 9.1.** *If $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ are vectors in $\mathbb{R}^n$, then the formula*

$$\langle u, v\rangle = u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

*defines $\langle u, v\rangle$ to be the Euclidean inner product on $\mathbb{R}^n$. The four inner product axioms hold (verify).*

The Euclidean inner product is the most important inner product on $\mathbb{R}^n$. However, there are various applications in which it is desirable to modify the Euclidean inner product by *weighting* its terms differently. More precisely, if

$$w_1, w_2, \cdots, w_n$$

are positive real numbers, which we shall call **weights**, and if $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ are vectors in $\mathbb{R}^n$, then it can be shown that the formula

$$\langle u, v\rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n$$

defines an inner product on $\mathbb{R}^n$, it is called the **weighted Euclidean inner product with weights** $w_1$**,** $w_2$**,** $\cdots$**,** $w_n$.

**Example 9.2.** *Let* $u = (u_1, u_2)$ *and* $v = (v_1, v_2)$ *be vectors in* $\mathbb{R}^2$*. Verify that the weighted Euclidean inner product*

$$\langle u, v\rangle = 3u_1v_1 + 2u_2v_2$$

*satisfies the four inner product axioms.*

**Solution:** Note first that if $u$ and $v$ are interchanged in this equation, the right side remains the same. Therefore,

$$\langle u, v\rangle = \langle v, u\rangle.$$

If $z = (z_1, z_2)$, then

$$\langle u+v, z\rangle = 3(u_1+v_1)z_1+2(u_2+v_2)z_2 = (3u_1z_1+2u_2z_2)+(3v_1z_1+2v_2z_2) = \langle u, z\rangle+\langle v, z\rangle$$

which establishes the second axiom.
Next,

$$\langle ku, v\rangle = 3(ku_1)v_1 + 2(ku_2)v_2 = k(3u_1v_1 + 2u_2v_2) = k\langle u, v\rangle$$

which establishes the third axiom.
Finally,

$$\langle v, v\rangle = 3v_1v_1 + 2v_2v_2 = 3v_1^2 + 2v_2^2$$

Obviously, $\langle v, v\rangle = 3v_1^2 + 2v_2^2 \geq 0$. Further, $\langle v, v\rangle = 3v_1^2 + 2v_2^2 = 0$ if and only if $v_1 = v_2 = 0$, that is, if and only if $v = (v_1, v_2) = 0$. Thus the fourth axiom is satisfied.

## 9.1 Length and Distance in Inner Product Spaces

Before discussing more examples of inner products, we shall pause to explain how inner products are used to introduce notions of length and distance in inner product spaces.

**Definition 9.2.** If $V$ is an inner product space, then the **norm** (or **length**) of a vector $u$ of $V$ is denoted by $\| u \|$ and is defined by

$$\| u \| = \sqrt{\langle u, u \rangle}.$$

The **distance** between two points (vectors) $u$ and $v$ is denoted by $d(u, v)$ and is defined by

$$d(u, v) = \| u - v \| .$$

If a vector has norm 1, then we say that it is a **unit vector**.

**Example 9.3.** *Let $u = (1, 0)$ and $v = (0, 1)$ be vectors in $\mathbb{R}^2$ with the Euclidean inner product, we have*

$$\| u \| = \sqrt{1^2 + 0^2} = 1$$

*and*

$$d(u, v) = \| u - v \| = \| (1, -1) \| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

**Example 9.4.** *If $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ are any two $2 \times 2$ matrices, then the following formula defines an inner product on $M_{2\times 2}$ (verify):*

$$\langle U, V \rangle = tr(U^T V) = tr(V^T U) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4.$$

*For example, if* $U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$ *then*

$$\langle U, V \rangle = 1(-1) + 2(0) + 3(3) + 4(2) = 16.$$

*The norm of a matrix* $U$ *relative to this inner product is*

$$\parallel U \parallel = \sqrt{\langle U, U \rangle} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} = \sqrt{1 + 4 + 9 + 16} = \sqrt{30}.$$

**Example 9.5.** *If* $p = a_0 + a_1 x + a_2 x^2$ *and* $q = b_0 + b_1 x + b_2 x^2$ *are any two vectors in* $P_2$*, then the following formula defines an inner product on* $P_2$ *(verify):*

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2.$$

*The norm of the polynomial* $p$ *relative to this inner product is*

$$\parallel p \parallel = \sqrt{\langle p, p \rangle} = \sqrt{a_0^2 + a_1^2 + a_2^2}.$$

The following theorem lists some basic algebraic properties of inner products.

**Theorem 9.6.** *If* $u$*,* $v$ *and* $w$ *are vectors in a real inner product space, and* $k$ *is any scalar, then*

**(a).** $\langle 0, v \rangle = \langle v, 0 \rangle = 0.$

**(b).** $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle.$

**(c).** $\langle u, kv \rangle = k\langle u, v \rangle.$

**(d).** $\langle u - v, w \rangle = \langle u, w \rangle - \langle v, w \rangle.$

**(e).** $\langle u, v - w\rangle = \langle u, v\rangle - \langle u, w\rangle$

*Proof.* We shall prove part (b) and leave the proofs of the remaining parts as exercises.

$$\langle u, v + w\rangle = \langle v + w, u\rangle = \langle v, u\rangle + \langle w, u\rangle = \langle u, v\rangle + \langle u, w\rangle.$$

□

**Theorem 9.7.** ***(Cauchy-Schwarz Inequality)*** *Let $V$ be an inner product space with inner product $\langle \cdot, \cdot\rangle$. Then*

$$|\langle u, v\rangle| \leqslant \| u \| \cdot \| v \|$$

*for all $u, v \in V$.*

*Proof.* If $v = 0$, then we see

$$\langle u, v\rangle = \langle u, 0\rangle = \langle u, 0 \cdot 0\rangle = 0\langle u, 0\rangle = 0.$$

Hence

$$|\langle u, v\rangle| = 0 = \| u \| \cdot \| v \|$$

as

$$\| v \| = 0.$$

In the remainder of the proof we assume $v \neq 0$. Let $\alpha$ be a scalar, put $w = u + \alpha v$ and expand $\langle w, w\rangle$:

$$
\begin{aligned}
0 \leqslant \langle w, w\rangle &= \langle u+\alpha v, u+\alpha v\rangle \\
&= \langle u, u\rangle + \alpha\langle v, u\rangle + \alpha\langle u, v\rangle + \alpha\alpha\langle v, v\rangle \\
&= \| u \|^2 + \alpha\langle u, v\rangle + \alpha\langle u, v\rangle + |\alpha|^2 \| v \|^2 .
\end{aligned}
$$

Now take $\alpha = \dfrac{-\langle u, u\rangle}{\| v \|^2}$. We deduce

$$
\begin{aligned}
0 &\leqslant \| u \|^2 - \frac{\langle u, v\rangle \cdot \langle u, v\rangle}{\| v \|^2} - \frac{\langle u, v\rangle \cdot \langle u, v\rangle}{\| v \|^2} + \frac{|\langle u, v\rangle|^2}{\| v \|^4} \| v \|^2 \\
&= \| u \|^2 - \frac{|\langle u, v\rangle|^2}{\| v \|^2},
\end{aligned}
$$

so

$$
|\langle u, v\rangle|^2 \leqslant \| u \|^2 \cdot \| v \|^2
$$

and taking square roots gives the result. $\square$

**Theorem 9.8.** *(**Triangle Inequality**) Let $V$ be an inner product space. Then*

$$
\| u+v \| \leqslant \| u \| + \| v \|
$$

*for all $u, v \in V$.*

*Proof.*

$$\begin{aligned}
\| u+v \|^2 &= \langle u+v, u+v \rangle \\
&= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle \\
&= \| u \|^2 + \langle u,v \rangle + \langle u,v \rangle + \| v \|^2 \\
&= \| u \|^2 + 2Re\langle u,v \rangle + \| v \|^2 \\
&\leqslant \| u \|^2 + 2|\langle u,v \rangle| + \| v \|^2 \\
&\leqslant \| u \|^2 + 2 \| u \| \cdot \| v \| + \| v \|^2 \quad \text{( by Cauchy-Schwarz)} \\
&= (\| u \| + \| v \|)^2
\end{aligned}$$

and taking square roots gives the result. □

## 9.2 Angle and Orthogonality in Inner Product Spaces

In this section we shall define the notions of an angle and Orthogonality between two vectors in an inner product space.

**Definition 9.3.** The **angle** $\theta$ between $u$ and $v$ is defined by the formula

$$cos\,\theta = \frac{\langle u,v \rangle}{\| u \| \| v \|} \quad 0 \leq \theta \leq \pi.$$

**Example 9.9.** *Let $\mathbb{R}^4$ have the Euclidean inner product. Find the cosine of the angle $\theta$ between the vectors $u = (4,3,1,-2)$ and $v = (-2,1,2,3)$.*

**Solution:** We leave the following as an exercise to verify that $\| u \| = \sqrt{30}$, $\| v \| = \sqrt{18}$ and $\langle u,v \rangle = -9$
so that

$$cos\,\theta = \frac{\langle u,v \rangle}{\| u \| \| v \|} = -\frac{9}{\sqrt{30}\sqrt{18}} = -\frac{3}{2\sqrt{15}}.$$

**Definition 9.4.** Two vectors $u$ and $v$ in an inner product space are called **orthogonal** if $\langle u, v\rangle = 0$.

**Example 9.10.** *If $M_{2\times 2}$ has the inner product of Example 9.4 in the preceding section, then the matrices* $U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ *and* $V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ *are orthogonal, since*

$$\langle U, V\rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0.$$

**Theorem 9.11.** *(**Generalized Theorem of Pythagoras**) If $u$ and $v$ are orthogonal vectors in an inner product space, then*

$$\| u + v \|^2 = \| u \|^2 + \| v \|^2 .$$

*Proof.* We leave the proof as an exercise. □

Followings are exercises for home work.

**Exercise 9.12.** *Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Show that the following are inner products on $\mathbb{R}^2$ by verifying that the inner product axioms hold.*

**(a).** $\langle u, v\rangle = 3u_1v_1 + 5u_2v_2$.

**(b).** $\langle u, v\rangle = 4u_1v_1 + u_2v_1 + u_1v_2 + 4u_2v_2$.

**Exercise 9.13.** *(**Parallelogram equality**) Let $V$ be an inner product space. Show that*

$$\| u + v \|^2 + \| u - v \|^2 = 2(\| u \|^2 + \| v \|^2).$$

*for all $u, v \in V$.*

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